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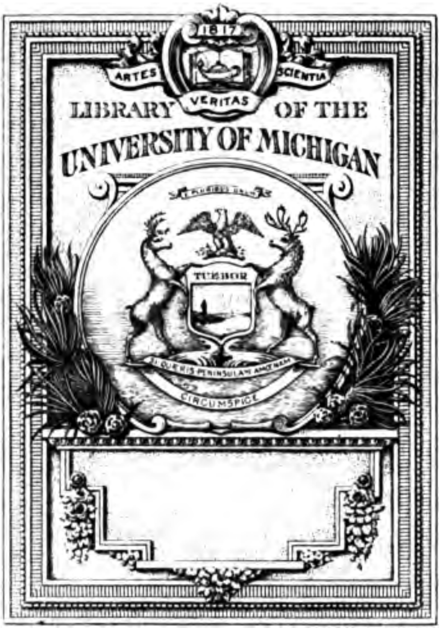
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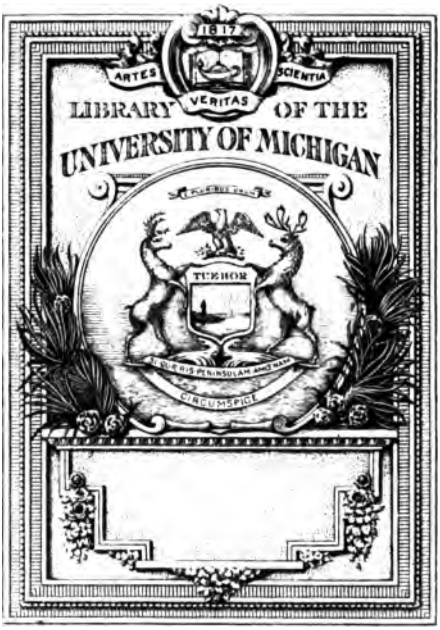

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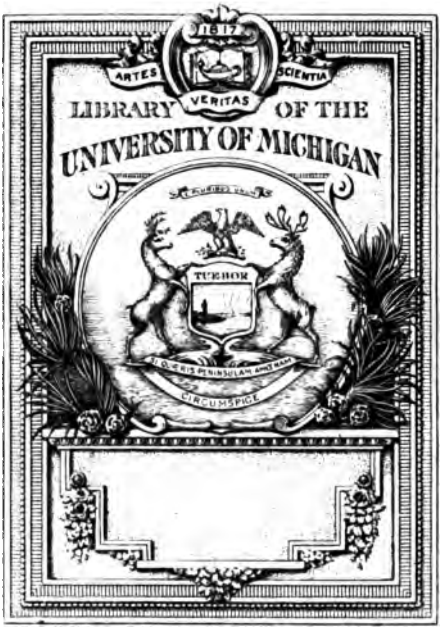
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ELEMENTS

OF

GEOMETRY;

CONTAINING

THE FIRST SIX BOOKS OF EUCLID,

WITH

TWO BOOKS

ON THE GEOMETRY OF SOLIDS.

TO WHICH ARE ADDED,

ELEMENTS OF

PLANE AND SPHERICAL

TRIGONOMETRY.

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PREFACE.

IT is a remarkable fact in the hiſtory of ſcience, that the oldeſt book of Elementary Geometry is ſtill conſidered as the beſt; and that the writings of Euclid, at the diſtance of two thouſand years, continue to form the moſt approved introduction to the mathematical ſciences. This peculiar diſtinction the Greek geometer owes to the elegance and correctneſs of his demonſtrations, added to an arrangement moſt happily contrived for the purpoſes of inſtruction; advantages which, when they reach a certain eminence, ſecure the works of an author from being forgotten, more effectually than even originality of invention. In paſſing, however, through the hands of the ancient editors, during the decline of ſcience, the excellence of his writings had been conſiderably obſcured, and much ſkill and learning have been employed by the modern mathematicians to deliver them from blemiſhes, with which, it is certain, that they were not originally marked. Of theſe mathematicians, Dr SIMSON, as he may be accounted the laſt, has alſo been the moſt ſucceſsful, and has left very little room for the ingenuity of future editors to be exerciſed in, either by amending the text of EUCLID, or by improving the tranſlations from it.

Such being the merits of Dr SIMSON's edition, and the reception it has met with having been every way ſuited to them, the work now offered to the public will perhaps appear unneceſſary. And indeed, if that geometer, had written with a view of accommodating the Elements of EUCLID to the preſent ſtate of the mathematical ſciences, it is not likely that any thing new in Elementary Geometry would have been ſoon attempted. But his deſign was different; it was his object to reſtore the writings of EUCLID to their original perfection, and to give them to modern Europe as nearly as poſſible in the ſtate wherein they made their firſt appearance in ancient Greece. For this undertaking no body could be better qualified than Dr SIMSON; who, to an accurate knowledge of the learned languages, and a moſt indefatigable ſpirit of reſearch, added a profound ſkill in the ancient Geometry, and an admiration of it almoſt enthuſiaſtic. Accordingly, he not only reſtored the text of EUCLID wherever it had been corrupted, but in ſome caſes removed imperfections that probably belonged to the original work; though his extreme partiality for his author never permitted him to ſuppoſe, that this was an honour that could fall to the ſhare either of himſelf, or of any other of the moderns.

But, after all this was accompliſhed, ſomething ſtill remained to be done, ſince, notwithſtanding the acknowledged excellence of Euclid's Elements, it could not be doubted, that ſome alterations might be made upon them, that would accommodate them better to a ſtate of the mathematical ſciences, ſo much more improved and extended than at any former period. This accordingly is the object of the edition now offered to the public, which is intended not ſo much to give to the writings of Euclid the form which they originally had, as that which may at preſent render them moſt uſeful.

One of the alterations that has been made with this view, reſpects the Doctrine of Proportion, the method of treating which, as it is laid down in the fifth of Euclid, has great advantages, accompanied with conſiderable defects; of which, however, it muſt be obſerved, that the advantages are eſſential to it, and the defects only accidental. To explain the nature of the former, requires a more minute examination than is ſuited to this place, and which muſt, therefore, be reſerved for the notes; and, in the mean time, it may be ſufficient to remark, that no definition of proportionals, except that of EUCLID, has ever been given, from which their properties can be deduced by reaſonings, which, at the ſame time that they are perfectly rigorous, are alſo ſimple and direct. As to the defects, on the other hand, the prolixneſs and obſcurity, that have ſo often been complained of in this book, they ſeem to ariſe entirely from the nature of the language; for, in mathematics, common language can ſeldom be applied, without much tedíouſneſs and circumlocution, in reaſoning about the relations of ſuch things as cannot be repreſented by means of diagrams, which happens here, where the ſubject treated of is magnitude in general. It is plain, therefore, that the conciſe language of Algebra is directly calculated to remedy this inconvenience; and ſuch a one I have, accordingly, endeavoured to introduce, in the ſimpleſt form, and without changing at all the nature of the reaſoning, or departing in any thing from the rigour of geometrical demonſtration. By this contrivance the ſteps of the reaſoning which were before ſo far ſeparated, are brought near to one another, and the force of the whole is ſo clearly and directly perceived, that I am perſuaded no more difficulty will be found in underſtanding the propoſitions of the fifth Book, than of any other of the Elements.

A few changes have alſo been made in the enunciations of this book, chiefly in thoſe of the ſubſidiary propoſitions which Euclid introduced for the ſake of the reſt; they are expreſſed here in the manner that ſeemed beſt adapted to the new notation.

The alterations above mentioned are the moſt material that have been attempted on the books of Euclid. There are, however, a few others, which, though leſs conſiderable, it is hoped, may in ſome degree facilitate the underſtanding of them. Such are thoſe made on the definitions in the firſt Book, and particularly on that of a ſtraight line. A new Axiom is alſo introduced in the room of the 12th, for the purpoſe of demonſtrating more eaſily ſome of the properties of parallel lines. In the third Book, the remarks concerning the angles made by a ſtraight line, and the circumference of a circle, are left out, as tending to perplex one who has advanced no farther than the elements of the ſcience. The 27th, 28th and 29th of the 6th are changed for eaſier and more ſimple propoſitions, which do not materially differ from them, and which anſwer exactly the ſame purpoſe. Some propoſitions alſo have been added; but, for a fuller detail concerning theſe changes, I muſt refer to the notes, in which ſeveral of the more difficult, or more intereſting ſubjects of Elementary Geometry are treated at conſiderable length.

Thus much for the part of the Elements that treats of Plane Figures. With reſpect to the Geometry of Solids, I have departed from Euclid altogether, with a view of rendering it both ſhorter and more comprehenſive. This, however, is not attempted by introducing a mode of reaſoning looſer or leſs rigorous than that of the Greek geometer; for this would be to pay too dear even for the time that might thereby be ſaved; but it is done chiefly by laying aſide a certain rule, which, though it be not eſſential to the accuracy of demonſtration, Euclid has thought it proper, as much as poſſible, to obſerve.

The rule referred to, is one which regulates the arrangement of Euclid's propoſitions through the whole of the Elements, *viz.* That in the demonſtration of a theorem he never ſuppoſes any thing to be done, as any line to be drawn, or any figure to be conſtructed, the manner of doing which he has not previouſly explained. Now, the only uſe of this rule is to prevent the admiſſion of impoſſible or contradictory ſuppoſitions, which no doubt might lead into error; and it is a rule well calculated to anſwer that end; as it does not allow the exiſtence of any thing to be ſuppoſed, unleſs the thing itſelf be actually exhibited. But it is not always neceſſary to make uſe of this defence, for the exiſtence of many things is obviouſly poſſible, and far enough from implying a contradiction, where the method of actually exhibiting them may be altogether unknown. Thus, it is plain, that on any given figure as a baſe, a ſolid may be conſtituted, or conceived to exiſt, equal to a given ſolid, (becauſe a ſolid, whatever be its baſe, as its height may be indefinitely varied, is capable of all degrees of magnitude, from nothing upwards), and yet, it may in many caſes be a problem of extreme difficulty to aſſign the height of ſuch a ſolid, and actually to exhibit it. Now, this very ſuppoſition is one of thoſe, by the introduction of which, the Geometry of Solids is much ſhortened, while all the real accuracy of the demonſtrations is preſerved; and therefore, to follow, as EUCLID has done, the rule that excludes this, and ſuch like hypotheſes, is to create artificial difficulties, and to embarraſs geometrical inveſtigation with more obſtacles than the nature of things has thrown in its way. It is a rule, too, which cannot always be followed, and from which even EUCLID himſelf has been forced to depart, in more than one inſtance.

In the two Books, therefore, on the Properties of Solids, that I now offer to the public, though

I have followed EUCLID very cloſely in the ſimpler parts, I have no where ſought to ſubject the demonſtrations to ſuch a law as the foregoing, and have never heſitated to admit the exiſtence of ſuch ſolids, or ſuch lines as are evidently poſſible, though the manner of actually deſcribing them may not have been explained. In this way alſo, I have been enabled to offer that very refined artifice in geometrical reaſoning, to which we give the name of the Method of Exhauſtions, under a much ſimpler form than it appears in the 12th of EUCLID; and the ſpirit of it may, I think, be beſt learned when it is diſengaged from every thing not eſſential to it. That this method may be the better underſtood, and becauſe the demonſtrations that require it are, no doubt, the moſt difficult in the Elements, they are all conducted as nearly as poſſible in the ſame way through the different ſolids, from the pyramid to the ſphere. The compariſon of this laſt ſolid with the cylinder concludes the eight Book, and is a propoſition that may not improperly be conſidered as terminating the elementary part of Geometry.

In the beginning of the Book juſt mentioned, I have treated pretty fully of the rectification and quadrature of the Circle, ſubjects that are often omitted altogether in works of this kind. They are omitted, however, as I conceive, without any good reaſon, becauſe, to meaſure the length of the ſimpleſt of all the curves which Geometry treats of, and the ſpace contained within it, are problems that certainly belong to the elements of the ſcience, eſpecially as they are not more difficult than other propoſitions where the method of exhauſtions is employed. When I ſpeak of the rectification of the circle, or of meaſuring the length of the circumference, I muſt not be ſuppoſed to mean, that a ſtraight line is to be made equal to the circumference *exactly*, a problem which, as is well known, Geometry has never been able to reſolve: All that is propoſed is, to determine two ſtraight lines that differ very little from one another, not more, for inſtance, than the four hundred and ninety-ſeventh part of the diameter of the circle, and of which the one is demonſtrated to be greater than the circumference of that circle, and the other to be leſs. In the ſame manner, the quadrature of the circle is performed only by approximation, or by finding two rectangles, nearly equal to one another, the one of them greater, and the other leſs than the ſpace contained within the circle.

The Data of EUCLID has been annexed to ſeveral editions of the Elements, and particularly to Dr SIMSON's, but in this it is omitted altogether. It is omitted, however, not from any opinion of its being in itſelf uſeleſs, but becauſe it does not belong to this place, and is not often read by beginners. It contains the rudiments of what is properly called the Geometrical Analyſis, and has itſelf an analytical form; and, for theſe reaſons, I would willingly reſerve it, or rather a compend of it, for a work on that analyſis, which I have long meditated.

Plane and Spherical Trigonometry, on the other hand, make a part of this volume, becauſe, in every courſe of mathematical ſtudies, that is directed toward uſeful purpoſes, theſe two branches neceſſarily come after the Elements. In explaining the elements of ſuch ſciences, there is not much new that can be attempted, or that will be expected by the intelligent reader. Except, perhaps, ſome new demonſtrations, and ſome changes in the arrangement, theſe two treatiſes have, accordingly, no novelty to boaſt of. The Plane Trigonometry, though pretty full, is ſo divided, that the part of it that is barely ſufficient for the reſolution of Triangles, may be eaſily taught by itſelf. In a ſcholium the method of conſtructing the trigonometrical Tables is explained, and a demonſtration is added of the properties of the ſines and co-ſines of the ſums and differences of arches, which are the foundation of thoſe new applications of Trigonometry that have been introduced with ſo much advantage into the higher Geometry.

In the Spherical Trigonometry, the rules for preventing the ambiguity of the ſolutions, whereever it can be prevented, have been particularly attended to; and I have availed myſelf as much as poſſible of that excellent abſtract of the rules of this ſcience, which Dr MASKELYNE has prefixed to the new tables of Logarithms.

It has been objected to many of the writers on Elementary Geometry, and particularly to EUCLID, that they have been at great pains to prove the truth of many ſimple propoſitions, which every body is ready to admit, without any demonſtration, and thus take up the time, and fatigue the attention of the ſtudent, to no purpoſe. To this objection alſo, if there be any force in it, the preſent treatiſe is certainly as much expoſed as any other, for, of all the alterations that may be made in the Elements, the laſt I ſhould think of, is to conſider any thing as ſelf-evident that admits of demonſtration. Indeed, thoſe who make the objection juſt ſtated, do not ſeem to have reflected ſufficiently on the end of Mathematical Demonſtration, which is not only to prove the truth of a certain propoſition, but to ſhew its neceſſary connection with other propoſitions, and its dependence on them. The truths of Geometry are all neceſſarily connected with one another, and the ſyſtem of ſuch truths can never be rightly explained, unleſs that connection be accurately traced, wherever it exiſts. It is upon this that the beauty and peculiar excellence of the mathematical ſciences depend; it is this that prevents any one truth from being ſingle and inſulated, and connects the different parts ſo firmly, that they muſt all ſtand, or all fall together. The demonſtration, therefore, even of an obvious propoſition, anſwers the purpoſe of connecting that propoſition with others, and aſcertaining its place in the general ſyſtem of mathematical truth. If, for example, it be alleged, that it is needleſs to demonſtrate that any two ſides of a triangle are greater than the third; it may be replied, that this is no doubt a truth, which, without proof, moſt men will be inclined to admit; but, are we for that reaſon to account it of no conſequence to know what the propoſitions are, which would ceaſe to be true if this propoſition were ſuppoſed to be falſe? Is it not uſeful to know, that unleſs it be true, that any two ſides of a triangle are greater than the third, neither could it be true, that the greater ſide of every triangle is oppoſite to the greater angle, nor that the equal ſides are oppoſite to equal angles, nor, laſtly, that things equal to the ſame thing are equal to one another? By a ſcientific mind this information will not be thought lightly of; and it is exactly that which we receive from Euclid's demonſtration.

To all this it may be added, that the mind, eſpecially when beginning to ſtudy the art of reaſoning, cannot be employed to greater advantage than in analyſing thoſe judgments, which, though they appear ſimple, are in reality complex, and capable of being diſtinguiſhed into parts. No progreſs in aſcending higher can be expected till a regular habit of demonſtration is thus acquired; and I ſhould greatly ſuſpect, that he who has declined the trouble of tracing the connection between the propoſition already quoted, and thoſe that are below it, would never be very expert in tracing its connection with thoſe that are above it; and that, as he had not been careful in laying the foundation, he would never be ſucceſsful in raiſing the ſuperſtructure.

College of Edinburgh,
Oct. 21. 1795.

ERRATA.

Book III. Definitions, for Def. XI. *read* Def. IX.

Book V. Prop. XXIV. line third of the Demonſtration, *read* Becauſe E : B : : F : D, by inverſion, B : E : : C : D.

P. 263. fourth line, for 2, *read* two.

ELEMENTS

OF

GEOMETRY.

BOOK I.

DEFINITIONS.

I.

"A Point is that which has poſition, but not magni- See Notes.
"tude."

II.

A line is length without breadth.

"Corollary. The extremities of a line are points; and
"the interſections of one line with another are alſo points."

III.

"Lines which cannot coincide in two points, without coin-
"ciding altogether, are called ſtraight lines.

"Cor. Hence two ſtraight lines cannot incloſe a ſpace. Nei-
"ther can two ſtraight lines have a common ſegment;
"that is, they cannot coincide in part, without coinciding
"altogether."

IV.

A ſuperficies is that which hath only length and breadth.

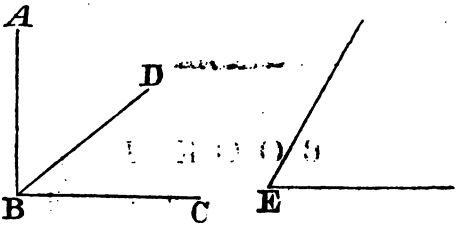
"Cor. The extremities of a ſuperficies are lines; and the
"interſections of one ſuperficies with another are alſo lines."

V.

A plane ſuperficies is that in which any two points being taken, the ſtraight line between them lies wholly in that ſuperficies.

VI.

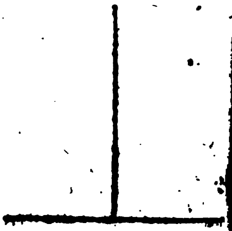
A plane rectilineal angle is the inclination of two ſtraight lines to one another, which meet together, but are not in the ſame ſtraight line.



N. B. ‘ When ſeveral angles are at one point B, any one of them is expreſſed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the ſtraight lines that contain the angle meet one another, is put between the other two letters, and one of theſe two is ſomewhere upon one of thoſe ſtraight lines, and the other upon the other line: Thus the angle which is contained by the ſtraight lines AB, CB, is named the angle ABC, or CBA; that which is contained by AB, BD is named the angle ABD, or DBA; and that which is contained by BD, CB is called the angle DBC, or CBD; but if there be only one angle at a point, it may be expreſſed by a letter placed at that point; as the angle at E.’

VII.

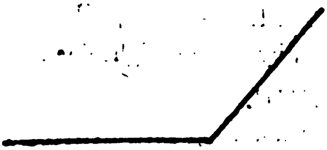
When a ſtraight line ſtanding on another ſtraight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the ſtraight line which ſtands on the other is called a perpendicular to it.

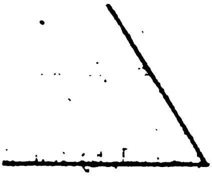


VIII.

VIII.

An obtuſe angle is that which is greater than a right angle.





IX.

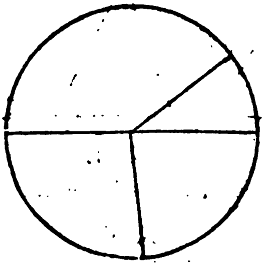
An acute angle is that which is leſs than a right angle.

X.

A figure is that which is incloſed by one or more boundaries.

XI.

A circle is a plane figure contained by one line, which is called the circumference, and is ſuch that all ſtraight lines drawn from a certain point within the figure to the circumference, are equal to one another.



XII.

And this point is called the centre of the circle.

XIII.

A diameter of a circle is a ſtraight line drawn through the centre, and terminated both ways by the circumference.

XIV.

A ſemicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XV.

Rectilineal figures are thoſe which are contained by ſtraight lines.

XVI.

Trilateral figures, or triangles, by three ſtraight lines.

XVII.

Quadrilateral, by four ſtraight lines.

XVIII.

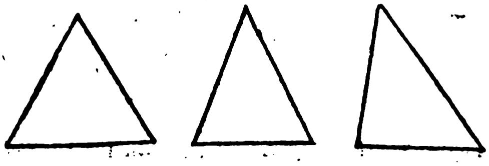
Multilateral figures, or polygons, by more than four ſtraight lines.

XIX.

Of three ſided figures, an equilateral triangle is that which has three equal ſides.

XX.

An iſoſceles triangle is that which has only two ſides equal.



XXI.

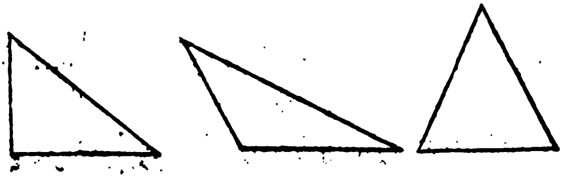
A ſcalene triangle, is that which has three unequal ſides.

XXII.

A right angled triangle, is that which has a right angle.

XXIII.

An obtuſe angled triangle, is that which has an obuſe angle.

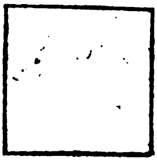


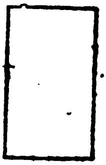
XXIV.

An acute angled triangle, is that which has three acute angles.

XXV.

Of four ſided figures, a ſquare is that which has all its ſides equal, and all its angles right angles.



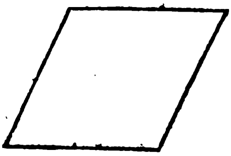


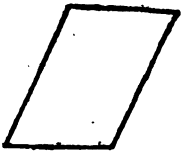
XXVI.

An oblong, is that which has all its angles right angles, but has not all its ſides equal.

XXVII.

A rhombus, is that which has all its ſides equal, but its angles are not right angles.





XXVIII.

A rhomboid, is that which has its oppoſite ſides equal to one another, but all its ſides are not equal, nor its angles right angles.

XXIX.

All other four ſided figures beſides theſe, are called Trapeziums.

XXX.

XXX.

Parallel ſtraight lines, are ſuch as are in the ſame plane, and which, being produced ever ſo far both ways, do not meet.

POSTULATES.

I.

LET it be granted that a ſtraight line may be drawn from any one point to any other point.

II.

That a terminated ſtraight line may be produced to any length in a ſtraight line.

III.

And that a circle may be deſcribed from any centre, at any diſtance from that centre.

AXIOMS.

I.

THINGS which are equal to the ſame thing are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the ſame, are equal to one another.

VII.

Things which are halves of the ſame, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the ſame ſpace, are equal to one another.

IX.

The whole is greater than its part.

X.

All right angles are equal to one another.

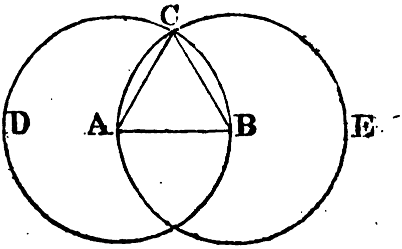
XI.

" Two ſtraight lines cannot be drawn through the ſame point,
" parallel to the ſame ſtraight line, without coinciding with
" one another.

PROPOSITION I. PROBLEM.

TO deſcribe an equilateral triangle upon a given finite ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe an equilateral triangle upon it.



From the centre A, at the diſtance AB, deſcribe [a] the circle BCD, and from the centre B, at the diſtance BA, deſcribe the circle ACE; and from the point C, in which the circles cut one another, draw the ſtraight lines [b] CA, CB to the points A, B: ABC ſhall be an equilateral triangle.

[a] 3. Poſtulate.

[b] 1. Poſt.

Becauſe the point A is the centre of the circle BCD, AC is equal [c] to AB; and becauſe the point B is the centre of the circle ACE, BC is equal to BA: But it has been proved that CA is equal to AB; therefore CA, CB are each of them equal to AB; but things which are equal to the ſame are equal to one another [d]; therefore CA is equal to CB; wherefore CA, AB, BC are equal to one another; and the triangle ABC is therefore equilateral, and it is deſcribed upon the given ſtraight line AB. Which was required to be done.

[c] 11. Definition.

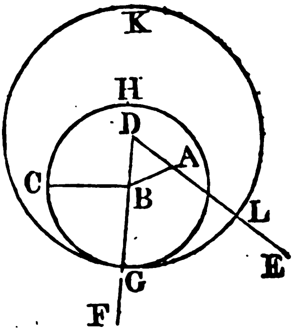
[d] 1. Axiom.

PROP. II. PROB.

FROM a given point to draw a ſtraight line equal to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw from the point A a ſtraight line equal to BC.

From



From the point A to B draw [a] the ſtraight line AB; and upon it deſcribe [b] the equilateral triangle DAB, and produce [c] the ſtraight lines DA, DB, to E and F; from the centre B, at the diſtance BC, deſcribe [d] the circle CGH, and from the centre D, at the diſtance DG, deſcribe the circle GKL. AL ſhall be equal to BC.

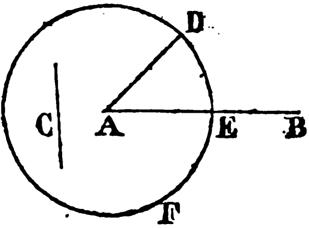
a 1 Poſt.
b 1. 1.
c 2. Poſt.
d 3. Poſt.

Becauſe the point B is the centre of the circle CGH, BC is equal [e] to BG; and becauſe D is the centre of the circle GKL, DL is equal to DG, and DA, DB, parts of them, are equal; therefore the remainder AL is equal to the remainder [f] BG: But it has been ſhewn, that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the ſame are equal to one another; therefore the ſtraight line AL is equal to BC. Wherefore from the given point A a ſtraight line AL has been drawn equal to the given ſtraight line BC. Which was to be done.

e 11. Def.
f 3. Ax.

PROP. III. PROB.

FROM the greater of two given ſtraight lines to cut off a part equal to the leſs.



Let AB and C be the two given ſtraight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the leſs.

From the point A draw [a] the ſtraight line AD equal to C; and from the centre A, and at the diſtance AD, deſcribe [b] the circle DEF; and becauſe A is the centre of the circle DEF, AE ſhall be equal to AD; but the ſtraight line C is likewiſe equal to AD; whence AE and C are each of them equal to AD; wherefore the ſtraight line AE is equal to [c] C, and from AB, the greater of two ſtraight lines, a part AE has been cut off equal to C the leſs. Which was to be done.

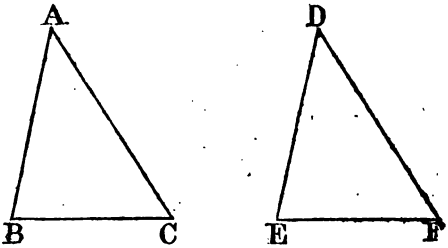
a 2. 1.
b 3. Poſt.
c 1. Ax.

PROP.

PROP. IV. THEOREM.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each; and have likewiſe the angles contained by thoſe ſides equal to one another; their baſes, or *third ſides*, ſhall be equal; and the two triangles ſhall be equal; and their other angles ſhall be equal, each to each, viz. thoſe to which the equal ſides are oppoſite.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; and the angle BAC equal to the angle EDF, the baſe BC ſhall be equal to the baſe EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal ſides are oppoſite, ſhall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.



For, if the triangle ABC be applied to the triangle DEF, ſo that the point A may be on D, and the ſtraight line AB upon DE; the point B ſhall coincide with the point E, becauſe AB is equal to DE; and AB coinciding with DE, AC ſhall coincide with DF, becauſe the angle BAC is equal to the angle EDF; wherefore alſo the point C ſhall coincide with the point F, becauſe AC is equal to DF: But the point B coincides with the point E; wherefore the baſe BC ſhall coincide with the baſe EF [a], and ſhall be equal to it. Therefore alſo the whole triangle ABC ſhall coincide with the whole triangle DEF, and be equal to it; and the remaining angles of the one ſhall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE. Therefore, if two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe the angles contained by thoſe ſides equal to one another, their baſes ſhall be equal, and the

[a] cor.def.3.

the triangles ſhall be equal, and their other angles, to which the equal ſides are oppoſite, ſhall be equal, each to each. Which was to be demonſtrated.

PROP. V. THEOR.

THE angles at the baſe of an Iſoſceles triangle are equal to one another; and, if the equal ſides be produced, the angles upon the other ſide of the baſe ſhall alſo be equal.

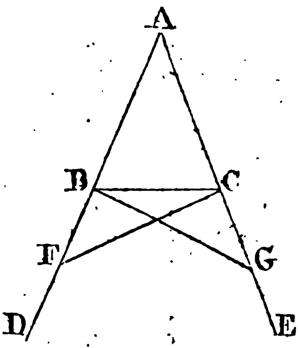
Let ABC be an iſoſceles triangle, of which the ſide AB is equal to AC, and let the ſtraight lines AB, AC be produced to D and E, the angle ABC ſhall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE the greater cut off AG equal [a] to AF, the leſs, and join FC, GB. a 3. 1.

Becauſe AF is equal to AG, and AB to AC, the two ſides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles, AFC, AGB; therefore the baſe FC is equal [b] to the baſe GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal [b] to the remaining angles of the other, each to each, to which the equal ſides are oppoſite, viz. the angle ACF to the angle ABG, and the angle AFC to the angle AGB: And becauſe the whole AF is equal to the whole AG, and the part AB to the part AC; the remainder BF ſhall be equal [c] to the remainder CG; and FC was proved to be equal to GB, therefore the two ſides BF, FC are equal to the two CG, GB, each to each; but the angle BFC is equal to the angle CGB; wherefore the triangles BFC, CGB are equal [b], and their remaining angles are equal, to which the equal ſides are oppoſite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG. Now, ſince it has been demonſtrated, that the whole angle ABG is equal to the whole

b 4. 1.

c 3. Ax.



Book I. whole ACF, and the part CBG to the part BCF, the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at the baſe of the triangle ABC: And it has alſo been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other ſide of the baſe. Therefore the angles at the baſe, &c. Q. E. D.

Corollary. Hence every equilateral triangle is alſo equiangular.

PROP. VI. THEOR.

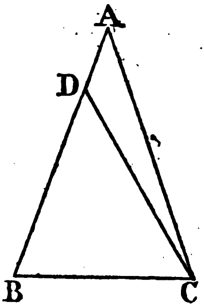
IF two angles of a triangle be equal to one another, the ſides which ſubtend, *or are oppoſite to*, thoſe angles, ſhall alſo be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the ſide AB is alſo equal to the ſide AC.

For, if AB be not equal to AC, one of them is greater than the other: Let AB be the greater, and from it cut [a] off DB equal to AC, the leſs, and join DC; therefore, becauſe in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two ſides DB, BC are equal to the two AC, CB, each to each; but the angle DBC is alſo equal to the angle ACB; therefore the baſe DC is equal to the baſe AB, and the triangle DBC is equal to the triangle [b] ACB, the leſs to the greater; which is abſurd. Therefore AB is not unequal to AC, that is, it is equal to it. Wherefore, if two angles, &c. Q. E. D.

[a] 3. 1.

[b] 4. 1.



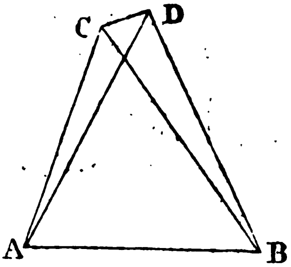
Cor. Hence every equiangular triangle is alſo equilateral.

PROP.

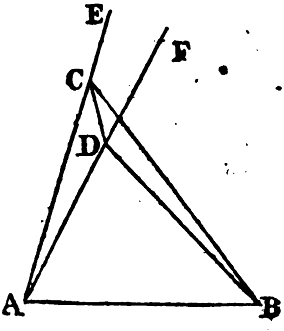
PROP. VII. THEOR.

UPON the ſame baſe, and on the ſame ſide of it, there cannot be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity, equal to one another. See N.

If it be poſſible, let there be two triangles ACB, ADB, upon the ſame baſe AB, and upon the ſame ſide of it, which have their ſides CA, DA, terminated in A equal to one another, and likewiſe their ſides CB, DB, terminated in B, equal to one another.



Join CD; then, in the caſe in which the vertex of each of the triangles is without the other triangle, becauſe AC is equal to AD, the angle ACD is equal [a] to the angle ADC: But the angle ACD is greater than the angle BCD; therefore the angle ADC is greater alſo than BCD; much more then is the angle BDC greater than the angle BCD. Again, becauſe CB is equal to DB, the angle BDC is equal [a] to the angle BCD; but it has been demonſtrated to be greater than it; which is impoſſible. a 5. 1.



But if one of the vertices, as D, be within the other triangle ACB; produce AC, AD to E, F; therefore, becauſe AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other ſide of the baſe CD are equal [a] to one another, but the angle ECD is greater than the angle BCD; wherefore the angle FDC is likewiſe greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, becauſe CB is equal to DB, the angle BDC is equal [a] to the angle BCD; but BDC has been

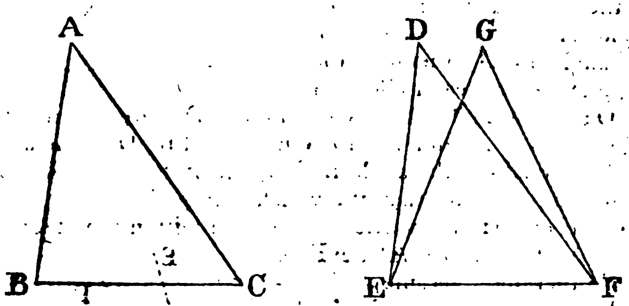
Book I. been proved to be greater than the ſame BCD; which is impoſſible. The caſe in which the vertex of one triangle is upon a ſide of the other, needs no demonſtration.

Therefore, upon the ſame baſe, and on the ſame ſide of it, there cannot be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity. Q. E. D.

PROP. VIII. THEOR.

If two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe their baſes equal; the angle which is contained by the two ſides of the one ſhall be equal to the angle contained by the two ſides of the other.

Let ABC, DEF be two triangles having the two ſides AB, AC, equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; and alſo the baſe BC equal to the baſe EF. The angle BAC is equal to the angle EDF.



For, if the triangle ABC be applied to the triangle DEF, ſo that the point B be on E, and the ſtraight line BC upon EF; the point C ſhall alſo coincide with the point F, becauſe BC is equal to EF: therefore BC coinciding with EF, BA and AC ſhall coincide with ED and DF; for, if BA, and CA do not coincide with ED, and FD, but have a different ſituation as EG and FG; then, upon the ſame baſe EF, and upon the ſame ſide of it, there can be two triangles EDF, EGF, that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe their ſides terminated in the other extremity: But this is impoſſible [a]; therefore, if the baſe BC coincides with the baſe EF, the ſides BA, AC cannot but coincide with the ſides ED, DF; wherefore likewiſe the angle BAC coincides with the angle EDF, and is equal [b] to it. Therefore if two triangles, &c. Q. E. D.

a 7. 1.
b 8. Ax.

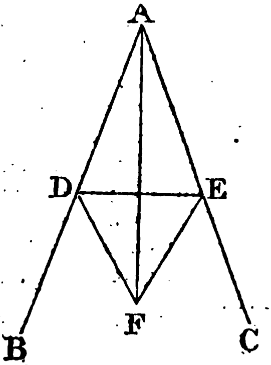
PROP. IX. PROB.

TO biſect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to biſect it.

Take any point D in AB, and from AC cut [a] off AE equal to AD; join DE, and upon it deſcribe [b] an equilateral triangle DEF; then join AF; the ſtraight line AF biſects the angle BAC.

a 3. 1.
b 1. 1.



Becauſe AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two ſides DA, AF, are equal to the two ſides EA, AF, each to each; but the baſe DF is alſo equal to the baſe EF; therefore the angle DAF is equal [c] to the angle EAF; wherefore the given rectilineal angle BAC is biſected by the ſtraight line AF. Which was to be done.

c 8. 1.

PROP. X. PROB.

TO biſect a given finite ſtraight line, that is, to divide it into two equal parts.

Let AB be the given ſtraight line; it is required to divide it into two equal parts.

Deſcribe [a] upon it an equilateral triangle ABC, and biſect [b] the angle ACB by the ſtraight line CD. AB is cut into two equal parts in the point D.

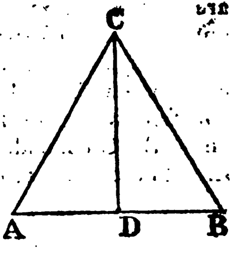
a 1. 1.
b 9. 1.

Becauſe

Book I.

Becauſe AC is equal to CB, and CD common to the two triangles ACD, BCD: the two ſides AC, CD are equal to the two BC, CD, each to each; but the angle ACD is alſo equal to the angle BCD; therfeore the baſe AD is equal to the baſe [c] DB, and the ſtraight line AB is divided into two equal parts in the point D. Which was to be done.

c 4. 1.

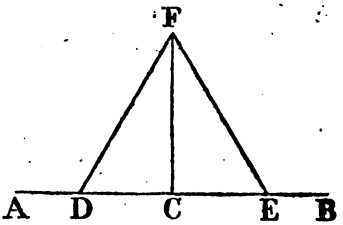


PROP. XI. PROB.

TO draw a ſtraight line at right angles to a given ſtraight line, from a given point in the ſame.

Let AB be a given ſtraight line, and C a point given in it; it is required to draw a ſtraight line from the point C at right angles to AB.

a 3. 1. Take any point D in AC, and [a] make CE equal to CD,
b 1. 1. and upon DE deſcribe [b] the equilateral triangle DFE, and join FC; the ſtraight line FC, drawn from the given point C, is at right angles to the given ſtraight line AB.



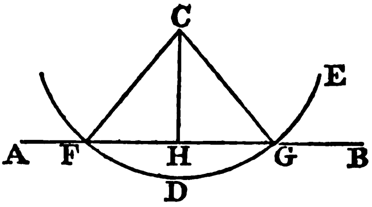
Becauſe DC is equal to CE, and FC common to the two triangles DCF, ECF, the two ſides DC, CF, are equal to the two EC, CF, each to each; but the baſe DF is alſo equal
c 8. 1. to the baſe EF; therefore the angle DCF is equal [c] to the angle ECF; and they are adjacent angles. But, when the adjacent angles which one ſtraight line makes with another ſtraight line are equal to one another, each of them is called a
d 7 def. right [d] angle; therefore each of the angles DCF, ECF, is a right angle. Wherefore, from the given point C, in the given ſtraight line AB, FC has been drawn at right angles to AB. Which was to be done.

PROP.

PROP. XII. PROB.

TO draw a ſtraight line perpendicular to a given ſtraight line of an unlimited length, from a given point without it.

Let AB be the given ſtraight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a ſtraight line perpendicular to AB from the point C.



Take any point D upon the other ſide of AB, and from the centre C, at the diſtance CD, deſcribe [a] the circle EGF meeting AB in F, G; and biſect [b] FG in H, and join CF, CH, CG; the ſtraight line CH, drawn from the given point C, is perpendicular to the given ſtraight line AB.

a 3. Poſt.
b 10. 1.

Becauſe FH is equal to HG, and HC common to the two triangles FHC, GHC, the two ſides FH, HC are equal to the two GH, HC, each to each; now the baſe CF is alſo equal [c] to the baſe CG; therefore the angle CHF is equal [d] to the angle CHG; and they are adjacent angles; but when a ſtraight line ſtanding on a ſtraight line makes the adjacent angles equal to one another, each of them is a right angle, and the ſtraight line which ſtands upon the other is called a perpendicular to it; therefore from the given point C a perpendicular CH has been drawn to the given ſtraight line AB. Which was to be done.

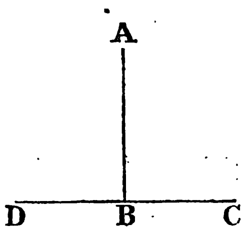
c 11. Def. 1.
d 8. 1.

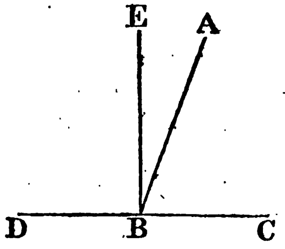
PROP. XIII. THEOR.

THE angles which one ſtraight line makes with another upon the one ſide of it, are either two right angles, or are together equal to two right angles.

Let the ſtraight line AB make with CD, upon one ſide of it, the angles CBA, ABD; theſe are either two right angles, or are together equal to two right angles.

For, if the angle CBA be equal to ABD, each of them is



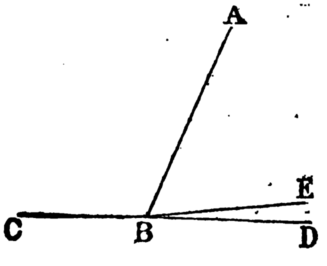


a def. 7. a right angle[a]; but, if not, from the point B draw BE at
b 11. 1. right angles[b] to CD; therefore the angles CBE, EBD are
two right angles; and becauſe CBE is equal to the two
angles CBA, ABE together, add the angle EBD to each of
c 2. Ax. theſe equals; therefore the angles CBE, EBD are equal[c] to
the three angles CBA, ABE, EBD. Again, becauſe the
angle DBA is equal to the two angles DBE, EBA, add to
theſe equals the angle ABC; therefore the angles DBA,
ABC are equal to the three angles DBE, EBA, ABC; but
the angles CBE, EBD have been demonſtrated to be equal to
the ſame three angles; and things that are equal to the ſame
d 1. Ax. are equal[d] to one another; therefore the angles CBE, EBD
are equal to the angles DBA, ABC; but CBE, EBD are two
right angles; therefore DBA, ABC are together equal to
two right angles. Wherefore, when a ſtraight line, &c.
Q. E. D.

PROP. XIV. THEOR.

IF, at a point in a ſtraight line, two other ſtraight lines, upon the oppoſite ſides of it, make the adjacent angles together equal to two right angles, theſe two ſtraight lines ſhall be in one and the ſame ſtraight line.

At the point B in the ſtraight line AB, let the two ſtraight lines BC, BD upon the oppoſite ſides of AB, make the adjancent angles ABC, ABD equal together to two right angles. BD is in the ſame ſtraight line with CB.



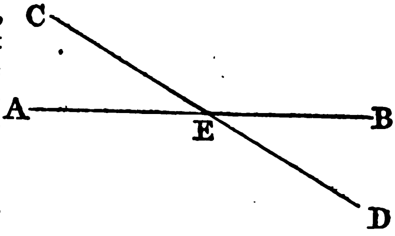
For, if BD be not in the ſame ſtraight line with CB,

let BE be in the ſame ſtraight line with it; therefore, becauſe the ſtraight line AB makes angles with the ſtraight line CBE, upon one ſide of it, the angles ABC, ABE are together equal [a] to two right angles; but the angles ABC, a 13. 1.
ABD are likewiſe together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: Take away the common angle ABC, and the remaining angle ABE is equal [b] to the remaining angle ABD, the leſs b 3. Ax.
to the greater, which is impoſſible; therefore BE is not in the ſame ſtraight line with BC. And, in like manner, it may be demonſtrated, that no other can be in the ſame ſtraight line with it but BD, which therefore is in the ſame ſtaight line with CB. Wherefore, if at a point, &c. Q. E. D.

PROP. XV. THEOR.

IF two ſtraight lines cut one another, the vertical, *or oppoſite* angles ſhall be equal.

Let the two ſtraight lines AB, CD cut one another in the point E; the angle AEC ſhall be equal to the angle DEB, and CEB to AED.



For the angles CEA, AED, which the ſtraight line AE makes with the ſtraight line CD, are together equal [a] to two right angles: a 13. 1.
and the angles AED, DEB, which the ſtraight line DE makes with the ſtraight line AB, are alſo together equal [a] to two right angles; therefore the two angles CEA, AED are equal to the two AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal [b] to the remaining angle DEB. In b 3 Ax.
the ſame manner it can be demonſtrated that the angles CEB, AED are equal. Therefore, if two ſtraight lines, &c. Q. E. D.

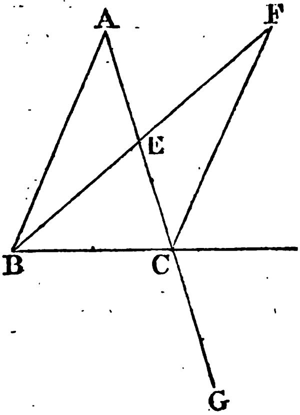
Cor. 1. From this it is manifeſt, that, if two ſtraight lines cut one another, the angles which they make at the point of their interſection, are together equal to four right angles.

Cor. 2. And hence, all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP. XVI. THEOR.

IF one ſide of a triangle be produced, the exter angle is greater than either of the interior opp ſite angles.

Let ABC be a triangle, and let its ſides BC be produ to D, the exterior angle ACD is greater than either of interior oppoſite angles CBA, BAC.

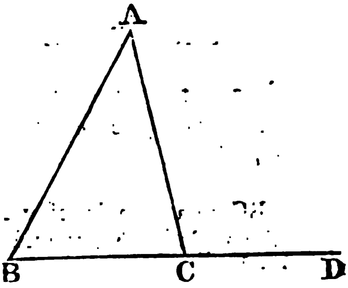


a 10. 1. Biſect[a] AC in E, join BE and produce it to F, and make EF equal to BE; join alſo FC, and produce AC to G.

Becauſe AE is equal to EC, and BE to EF; AE, EB are equal to CE, EF, each to each; and the angle AEB is b 15. 1. equal[b] to the angle CEF, becauſe they are oppoſite vertical angles; therefore the baſe c 4. 1. AB is equal[c] to the baſe CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remai angles, each to each, to which the equal ſides are oppoſ wherefore the angle BAE is equal to the angle ECF; the angle ECD is greater than the angle ECF; therefore angle ECD, that is ACD, is greater than BAE: In the ſ manner, if the ſide BC be biſected, it may be demonſtr d 15. 1. that the angle BCG, that is[d], the angle ACD, is greater t the angle ABC. Therefore, if one ſide, &c. Q. E. D.

PROP. XVII. THEOR.

ANY two angles of a triangle are together than two right angles.



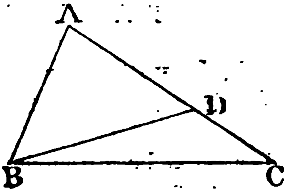
Let ABC be any triangle; y two of its angles toge-er are leſs than two right ngles.

Produce BC to D; and ecauſe ACD is the exterior ngle of the triangle ABC, ACD is greater[a] than the interior and oppoſite angle ABC; to each of theſe add he angle ACB; therefore he angles ACD, ACB are greater than the angles ABC, ACB; but ACD, ACB are together equal[b] to two right ngles; therefore the angles ABC, BCA are leſs than two ight angles. In like manner, it may be demonſtrated, that BAC, ACB, as alſo CAB, ABC, are leſs than two right ngles. Therefore, any two angles, &c. Q. E. D.

a 16. 1.
b 13. 1.

PROP. XVIII. THEOR.

THE greater ſide of every triangle has the greater angle oppoſite to it.



Let ABC be a triangle, of hich the ſide AC is greater ha the ſide AB; the angle ABC alſo greater than the angle CA.

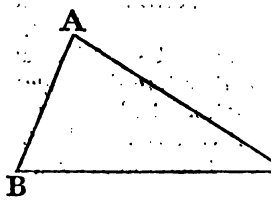
Becauſe AC is greater than B, make[a] AD equal to AB, d join BD; and becauſe ADB is the exterior angle of the iangle BDC, it is greater[b] than the interior and oppoſite ngle DCB; but ADB is equal[c] to ABD, becauſe the ſide B is equal to the ſide AD; therefore the angle ABD is kewiſe greater than the angle ACB; wherefore much more the angle ABC greater than ACB. Therefore the great-r ſide, &c. Q. E. D.

a 3. 1.
b 16. 1.
c 5. 1.

PROP. XIX. THEOR.

THE greater angle of every triangle is ſubte ed by the greater ſide, or *has the greater* *oppoſite to it.*

Let ABC be a triangle, of which the angle ABC is gr er than the angle BCA; the ſide AC is likewiſe greater t the ſide AB.

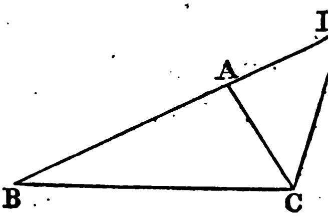


For, if it be not greater, AC muſt either be equal to AB, or leſs than it; it is not equal, becauſe then the angle ABC would
a 5. 1. be equal [a] to the angle ACB; but it is not; therefore AC is not equal to AB; neither is it leſs; becauſe then the angle
b 18. 1. ABC would be leſs [b] than the angle ACB; but it is n therefore the ſide AC is not leſs than AB; and it has b ſhewn that it is not equal to AB; therefore AC is gre than AB. Wherefore the greater angle, &c. Q. E. D.

PROP. XX. THEOR.

ANY two ſides of a triangle are together grea than the third ſide.

Let ABC be a triangle; any two ſides of it together greater than the third ſide, *viz.* the ſides BA, AC grea than the ſide BC; and AB, BC greater than AC; and I CA greater than AB.



Produce BA to the point
a 3. 1. D, and make [a] AD equal to AC; and join DC.

Becauſe DA is equal to AC, the angle ADC is like-
b 5. 1. wiſe equal [b] to ACD; but the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle ADC; and

ca

auſe the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater c ſide is oppoſite to the greater angle; therefore the ſide DB is greater than the ſide BC; but DB is equal to BA and AC; therefore the ſides BA, AC are greater than BC. In the ſame manner it may be demonſtrated, that the ſides AB, BC are greater than CA, and BC, CA greater than AB. Therefore any two ſides, &c. Q. E. D.

c 19. 1.

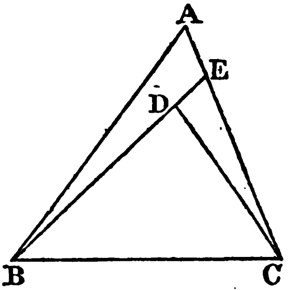
PROP. XXI. THEOR.

See N.

IF from the ends of the ſide of a triangle, there be drawn two ſtraight lines to a point within the triangle, theſe two lines ſhall be leſs than the other two ſides of the triangle, but ſhall contain a greater angle.

Let the two ſtraight lines BD, CD be drawn from B, C, the ends of the ſide BC of the triangle ABC, to the point D within it; BD and DC are leſs than the other two ſides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and becauſe two ſides of a triangle are greater than the third ſide, the two ſides BA, AE of the triangle ABE are greater than BE. To each of theſe add EC; therefore the ſides BA, AC are greater than BE, EC: Again, becauſe the two ſides CE, ED of the triangle CED are greater than CD, add DB to each of theſe; therefore the ſides CE, EB are greater than CD, DB; but it has been ſhewn that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.



Again, becauſe the exterior angle of a triangle is greater than the interior and oppoſite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the ſame reaſon, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonſtrated that the angle

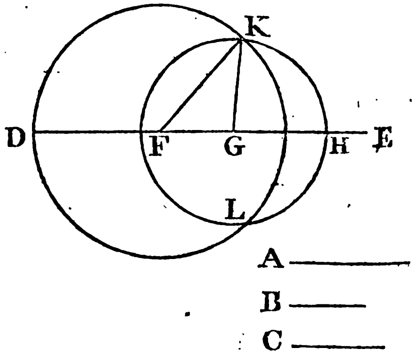
BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. Therefore, if from the ends of, &c. Q. E. D.

PROP. XXII. PROB.

TO make a triangle of which the ſides ſhall be equal to three given ſtraight lines; but any two whatever of theſe lines muſt be greater than the a 20. 1. third [a].

Let A, B, C be the three given ſtraight lines, of which any two whatever are greater than the third, viz. A and B greater than C; A and C greater than B; and B and C than A. It is required to make a triangle of which the ſides ſhall be equal to A, B, C, each to each.

Take a ſtraight line DE terminated at the point D, but unlimited towards E, a 3. 1. and make [a] DF equal to A, FG to B, and GH equal to C; and from the centre F, at the diſtance FD, b 3. Poſt. deſcribe [b] the circle DKL; and from the centre G, at the diſtance GH, deſcribe [b] another circle HLK; and join KF, KG; the triangle KFG has its ſides equal to the three ſtraight lines, A, B, C.

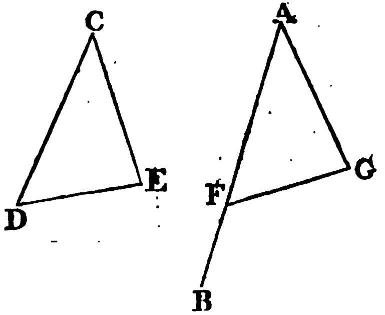


Becauſe the point F is the centre of the circle DKL, FD is c 11. Def. equal [c] to FK; but FD is equal to the ſtraight line A; therefore FK is equal to A: Again, becauſe G is the centre of the circle LKH, GH is equal [c] to GK; but GH is equal to C; therefore alſo GK is equal to C; and FG is equal to B; therefore the three ſtraight lines KF, FG, GK, are equal to the three A, B, C: And therefore the triangle KFG has its three ſides KF, FG, GK equal to the three given ſtraight lines, A, B, C. Which was to be done.

PROP. XXIII. PROB.

AT a given point in a given ſtraight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given ſtraight line AB, that ſhall be equal to the given rectilineal angle DCE.



Take in CD, CE any points D, E, and join DE; and make[a] the triangle AFG, the ſides of which ſhall be equal to the three ſtraight lines CD, DE, CE, ſo that CD be equal to AF, CE to AG, and DE to FG; and becauſe DC, CE are equal to FA, AG, each to each, and the baſe DE to the baſe FG; the angle DCE is equal[b] to the angle FAG. Therefore, at the given point A in the given ſtraight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

a 22. 1.

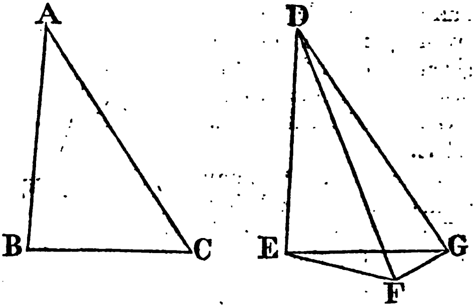
b 8. 1.

PROP. XXIV. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the angle contained by the two ſides of one of them greater than the angle contained by the two ſides equal to them, of the other; the baſe of that which has the greater angle ſhall be greater than the baſe of the other.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the baſe BC is alſo greater than the baſe EF.

Of the two ſides DE, DF, let DE be the ſide which is not greater than the other, and at the point D, in the ſtraight line
a 23. 1. DE, make [a] the angle EDG equal to the angle BAC; and
b 3. 1. make DG equal [b] to AC or DF, and join EG, GF.



Becauſe AB is equal to DE, and AC to DG, the two ſides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the baſe BC is
c. 4. 1. equal [c] to the baſe EG; and becauſe DG is equal to DF, the angle
d. 5. 1. DFG is equal [d] to the angle DGF; but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; and much more is the angle EFG greater than the angle EGF; and becauſe the angle EFG of the triangle
e. 19. 1. EFG is greater than its angle EGF, and that the greater [e] ſide is oppoſite to the greater angle; the ſide EG is therefore greater than the ſide EF; but EG is equal to BC; and therefore alſo BC is greater than EF. Therefore, if two triangles, &c. Q. E. D.

PROP. XXV. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the baſe of the one greater than the baſe of the other; the angle contained by the ſides of that which has the greater baſe, ſhall be greater than the angle contained by the ſides equal to them, of the other.

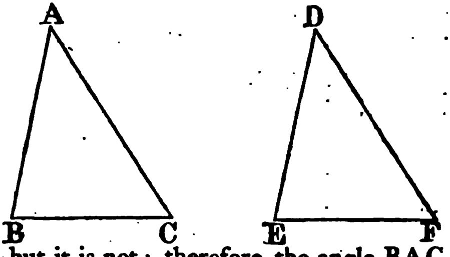
Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE DF, each to each, viz.

AB equal to DE, and AC to DF; but the baſe CB is greater than the baſe EF; the angle BAC is likewiſe greater than the angle EDF.

For, if it be not greater, it muſt either be equal to it, or leſs; but the angle BAC is not equal to the angle EDF, becauſe then the baſe BC would be equal [a] to EF; but it is not; therefore the angle BAC is not equal to the angle EDF; neither is it leſs; becauſe then the baſe BC would be leſs [b] than the baſe EF; but it is not; therefore the angle BAC is not leſs than the angle EDF; and it was ſhewn that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore, if two triangles, &c. Q. E. D.

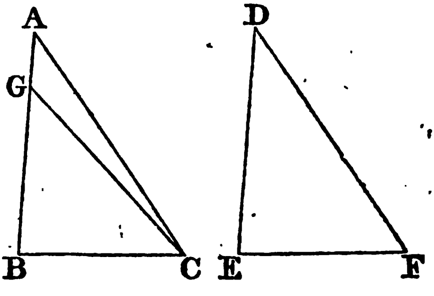
a 4. 1.

b 24. 1.



PROP. XXVI. THEOR.

IF two triangles have two angles of the one equal to two angles of the other, each to each; and one ſide equal to one ſide, viz. either the ſides adjacent to the equal angles, or the ſides oppoſite to the equal angles in each; then ſhall the other ſides be equal, each to each; and alſo the third angle of the one to the third angle of the other.

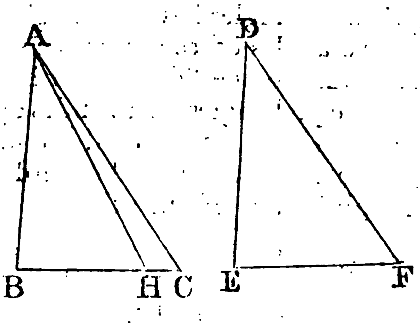


Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; alſo one ſide equal to one ſide; and firſt let thoſe ſides be equal which are adjacent to the angles that are equal in the two triangles, viz. BC to EF;

the other ſides ſhall be equal, each to each, viz. AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.

For, if AB be not equal to DE, one of them muſt be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; therefore, becauſe BG is equal to DE, and BC to EF, the two ſides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal
a 4. 1. to the angle DEF; therefore the baſe GC is equal [a] to the baſe DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal ſides are oppoſite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypotheſis, equal to the angle BCA; wherefore alſo the angle BCG is equal to the angle BCA, the leſs to the greater; which is impoſſible; therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; the baſe therefore AC is equal [a] to the baſe DF, and the third angle BAC to the third angle EDF.

Next, let the ſides which are oppoſite to equal angles in each triangle be equal to one another, viz. AB to DE; likewiſe in this caſe, the other ſides ſhall be equal, AC to DF, and BC to EF; and alſo the third angle BAC to the third EDF.



For, if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH; and becauſe BH is equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore the baſe AH is equal to the baſe DF, and the triangle ABH to the triangle DEF, and the other angles ſhall be equal, each to each, to which the equal ſides are oppoſite; therefore the angle BHA is equal to the angle

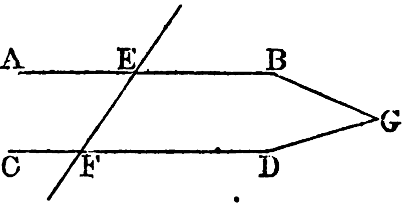
EFD; but EFD is equal to the angle BCA; therefore alſo the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and oppoſite angle BCA; which is impoſſible[b]; wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two AB, BC, are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the baſe AC is equal to the baſe DF, and the third angle BAC to the third angle EDF. Therefore, if two triangles, &c. Q. E. D. b 16. 1.

PROP. XXVII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the alternate angles equal to one another, theſe two ſtraight lines ſhall be parallel.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

For, if it be not parallel, AB and CD being produced ſhall meet either towards B, D, or towards A, C; let them be produced and meet towards B, D in the point G; therefore GEF is a triangle, and its exterior angle AEF is greater[a] than the interior and oppoſite angle EFG; but it is alſo equal to it, which is impoſſible; therefore, AB and CD being produced, do not meet towards B, D. In like manner it may be demonſtrated that they do not meet towards A, C; but thoſe ſtraight lines which meet neither way, though produced ever ſo far, are parallel[b] to one another. AB therefore is parallel to CD. Wherefore, if a ſtraight line, &c. Q. E. D. a 16. 1. b 30. Def.



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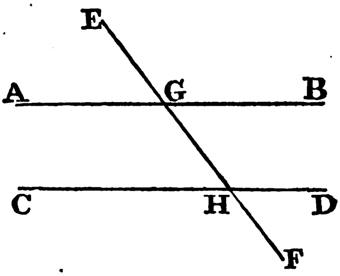
PROP. XXVIII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the exterior angle equal to the interior and oppoſite upon the ſame ſide of the line; or makes the interior angles upon the ſame ſide together equal to two right angles; the two ſtraight lines ſhall be parallel to one another.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD, make the exterior angle EGB equal to the interior and oppoſite angle GHD upon the ſame ſide; or make the interior angles on the ſame ſide BGH, GHD together equal to two right angles; AB is parallel to CD.



Becauſe the angle EGB is equal to the angle GHD, and the angle EGB equal[a] to the angle AGH, the angle AGH is equal to the angle GHD; and they are the alternate angles; therefore AB is parallel[b] to CD. Again, becauſe the angles BGH, GHD are equal[c] to two right angles; and that AGH, BGH, are alſo equal[d] to two right angles; the angles AGH, BGH are equal to the angles BGH, GHD: Take away the common angle BGH; therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles; therefore AB is parallel to CD. Wherefore, if a ſtraight line, &c. Q. E. D.

[a] 15. 1.
[b] 27. 1.
[c] By Hyp.
[d] 13. 1.

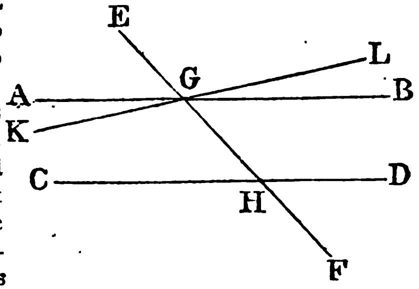
PROP. XXIX. THEOR.

See N.

IF a ſtraight line fall upon two parallel ſtraight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and oppoſite upon the ſame ſide; and likewiſe the two interior angles upon the ſame ſide together equal to two right angles.

Let the ſtraight line EF fall upon the parallel ſtraight lines AB, CD; the alternate angles AGH, GHD are equal to one another; and the exterior angle EGB is equal to the interior and oppoſite, upon the ſame ſide, GHD; and the two interior angles BGH, GHD upon the ſame ſide are together equal to two right angles.

For if AGH be not equal to GHD, let KG be drawn making the angle KGH equal to GHD, and produce KG to L; then KL will be parallel to CD[a]; but AB is alſo parallel to CD; therefore two ſtraight lines are drawn through the ſame point G, parallel to CD, and yet not coinciding with one another, which is impoſſible[b]. The angles AGH, GHD therefore are not unequal, that is, they are equal to one another. Now, the angle EGB is equal to AGH[c]; and AGH is proved to be equal to GHD; therefore EGB is likewiſe equal to GHD; add to each of theſe the angle BGH; therefore the angles EGB, BGH are equal to the angles BGH, GHD; but EGB, BGH are equal[d] to two right angles; therefore alſo BGH, GHD are equal to two right angles. Wherefore, if a ſtraight line, &c. Q. E. D.



a 27. 1.

b 11. Ax.

c 15. 1.

d 13. 1.

Cor. If two lines KL and CD make, with EF, the two angles KGH, GHC together leſs than two right angles, KG and GH will meet on the ſide of EF on which the two angles are that are leſs than two right angles.

For, if not, KL and CD are either parallel, or they meet on the other ſide of EF; but they are not parallel; for the angles KGH, GHC would then be equal to two right angles. Neither do they meet toward the points L and D; for the angles LGH, GHD would then be two angles of a triangle, and leſs than two right angles; but this is impoſſible; for the four angles KGH, HGL, CHG, GHD are together equal to four right angles[d], of which the two KGH, CHG are by ſuppoſition leſs than two right angles; therefore the other two, HGL, GHD are greater than two right angles. Therefore, ſince KL and CD are not parallel, and do not meet towards L and D, they will meet if produced towards K and C.

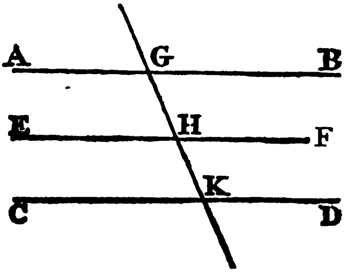
PROP.

PROP. XXX. THEOR.

STRAIGHT lines which are parallel to the ſame ſtraight line are parallel to one another.

Let AB, CD be each of them parallel to EF; AB is alſo parallel to CD.

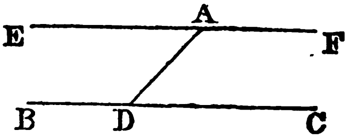
Let the ſtraight line GHK cut AB, EF, CD; and becauſe GHK cuts the parallel ſtraight lines AB, EF, the angle
a 29. 1. AGH is equal[a] to the angle GHF. Again, becauſe the ſtraight line GK cuts the parallel ſtraight lines EF, CD, the angle GHF is equal[a] to the angle GKD; and it was ſhewn that the angle AGK is equal to the angle GHF; therefore alſo AGK is equal to GKD; and they are alternate angles; therefore
b 27. 1. AB is parallel[b] to CD. Wherefore ſtraight lines, &c. Q. E. D.



PROP. XXXI. PROB.

TO draw a ſtraight line through a given point parallel to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw a ſtraight line through the point A, parallel to the ſtraight line BC.



In BC take any point D, and join AD; and at
a 23. 1. the point A, in the ſtraight line AD make[a] the angle DAE equal to the angle ADC; and produce the ſtraight line EA to F.

Becauſe the ſtraight line AD, which meets the two ſtraight lines BC, EF, makes the alternate angles EAD, ADC equal
b 27. 1. to one another, EF is parallel[b] to BC. Therefore the ſtraight

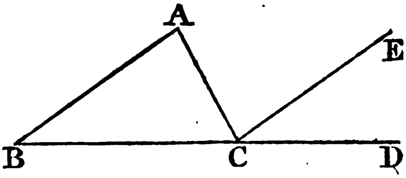
Book I.

line EAF is drawn through the given point A parallel to the given ſtraight line BC. Which was to be done.

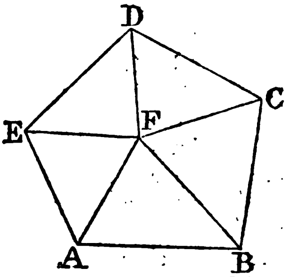
PROP. XXXII. THEOR.

IF a ſide of any triangle be produced, the exterior angle is equal to the two interior and oppoſite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its ſides BC be produced to D; the exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC and the three interior angles of the triangle, viz. ABC, BCA, CAB, are together equal to two right angles.



Through the point C draw CE parallel[a] to the ſtraight line AB; a 31. 1. and becauſe AB is parallel to CE and AC meets them, the alternate angles BAC, ACE are equal[b]. b 29. 1. Again, becauſe AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and oppoſite angle ABC; but the angle ACE was ſhown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC; to theſe angles add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal[c] to two c 13. 1. right angles; therefore alſo the angles CBA, BAC, ACB are equal to two right angles. Wherefore, if a ſide of a triangle, &c. Q. E. D.

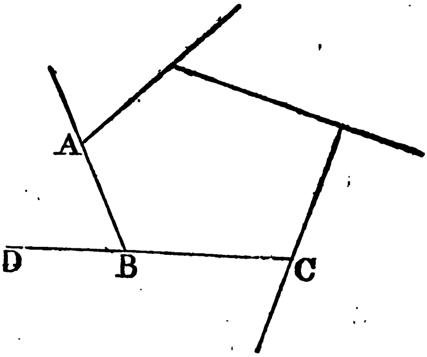


Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has ſides, by drawing ſtraight lines from a point F within the figure to each of its angles. And, by the preceding propoſition, all the angles of theſe triangles are equal to twice as many right angles as there are triangles, that is, as there are ſides of the figure; and the ſame angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles: that is [a], together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

a 2. Cor. 15. 1.

Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

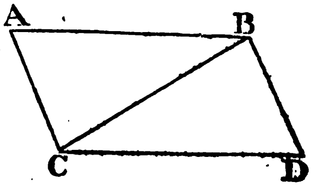


Becauſe every interior angle ABC, with its adjacent exterior ABD, is equal [b] to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are ſides of the figure; that is by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.

b 13. 1.

PROP. XXXIII. THEOR.

THE ſtraight lines which join the extremities of two equal and parallel ſtraight lines, towards the ſame parts, are alſo themſelves equal and parallel.



Let AB, CD be equal and parallel ſtraight lines, and joined towards the ſame parts by the ſtraight lines AC, BD; AC, BD are alſo equal and parallel.

Join BC; and becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal [a]; and becauſe

a 29. 1.

cauſe AB is equal to CD, and BC common to the two triangles ABC, DCB, the two ſides AB, BC are equal to the two DC, CB; and the angle ABC is equal to the angle BCD; therefore the baſe AC is equal[b] to the baſe BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles[b], each to each, to which the equal ſides are oppoſite; therefore the angle ACB is equal to the angle CBD; and becauſe the ſtraight line BC meets the two ſtraight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel[c] to BD; and it was ſhewn to be equal to it. Therefore, ſtraight lines, &c. Q. E. D.

b 4. 1.

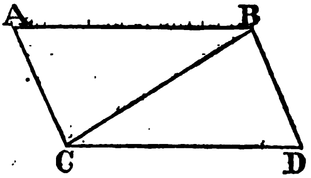
c 27. 1.

PROP. XXXIV. THEOR.

THE oppoſite ſides and angles of parallelograms are equal to one another, and the diameter biſects them, that is, divides them in two equal parts.

N. B. *A parallelogram is a four ſided figure, of which the oppoſite ſides are parallel; and the diameter is the ſtraight line joining two of its oppoſite angles.*

Let ACDB be a parallelogram, of which BC is a diameter; the oppoſite ſides and angles of the figure are equal to one another; and the diameter BC biſects it.



Becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal[a] to one another; and becauſe AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal[a] to one another; wherefore the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD in the other, each to each, and one ſide BC common to the two triangles, which is adjacent to their equal

a 29. 1.

angles; therefore their other ſides ſhall be equal, each to each, and the third angle of the one to the third angle of the other b, *viz.* the ſide AB to the ſide CD, and AC to BD, and the angle BAC equal to the angle BDC: And becauſe the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is equal to the whole angle ACD: And the angle BAC has been ſhown to be equal to the angle BDC; therefore the oppoſite ſides and angles of parallelograms are equal to one another: alſo, their diameter biſects them; for AB being equal to CD, and BC common, the two AB, BC are equal to the two DC, CB, each to each; and the angle ABC is equal to the angle BCD; therefore the triangle ABC is equal c to the triangle BCD, and the diameter BC divides the parallelogram ACDB into two equal parts. Q. E. D.

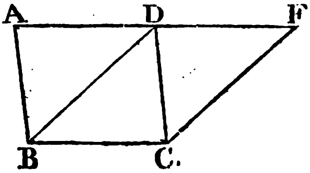
b 26. 1.
c 4. 1.

PROP. XXXV. THEOR.

PARALLELOGRAMS upon the ſame baſe and between the ſame parallels, are equal to one another.

See the 2d and 3d figures.

Let the parallelograms ABCD, EBCF be upon the ſame baſe BC, and between the ſame parallels AF, BC; the parallelogram ABCD ſhall be equal to the parallelogram EBCF.



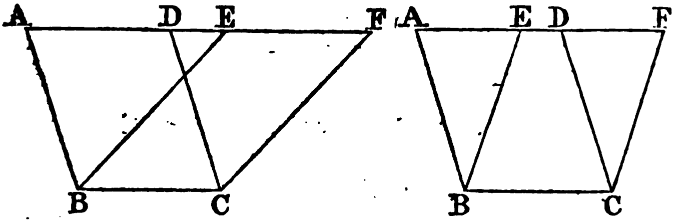
If the ſides AD, DF of the parallelograms ABCD, DBCF oppoſite to the baſe BC be terminated in the ſame point D; it is plain that each of the parallelograms is double a of the triangle BDC; and they are therefore equal to one another.

a 34. 1.

But, if the ſides AD, EF, oppoſite to the baſe BC of the parallelograms ABCD, EBCF, be not terminated in the ſame point; then, becauſe ABCD is a parallelogram, AD is equal a to BC; for the ſame reaſon EF is equal to BC; wherefore AD is equal b to EF; and DE is common; therefore the whole, or the remainder, AE is equal c to the whole, or the

b 1. Ax.
c 2. or 3. Ax.

the remainder DF; AB alſo is equal to DC; and the two



EA, AB are therefore equal to the two FD, DC, each to each; and the exterior angle FDC is equal [d] to the interior EAB, therefore the baſe EB is equal to the baſe FC, and the triangle EAB equal [e] to the triangle FDC; take the triangle FDC from the trapezium ABCF, and from the ſame trapezium take the triangle EAB; the remainders therefore are equal [f], that is, the parallelogram ABCD is equal to the parallelogram EBCF. Therefore, parallelograms upon the ſame baſe, &c. Q. E. D.

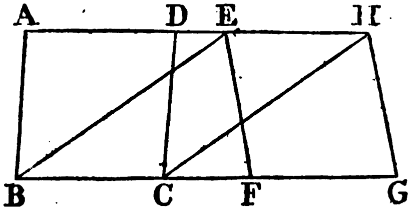
d

e 4.

f 3. Ax.

PROP. XXXVI. THEOR.

PARALLELOGRAMS upon equal baſes, and between the ſame parallels, are equal to one another.



Let ABCD, EFGH be parallelograms upon equal baſes BC, FG, and between the ſame parallels AH, BG; the parallelogram ABCD is equal to EFGH.

Join BE, CH; and becauſe BC is equal to FG, and FG to [a] EH, BC is equal to EH; and they are parallels, and joined towards the ſame parts by the ſtraight lines BE, CH: But ſtraight lines which join equal and parallel ſtraight lines towards the ſame parts, are themſelves equal and parallel [b]; therefore EB, CH are both equal and parallel, and EBCH is a parallelogram; and it is equal [c] to ABCD, becauſe it is upon the ſame baſe BC,

a 34. 1.

b 33. 1.

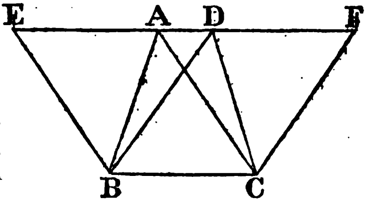
c 35. 1.

and between the ſame parallels BC, AD: For the like reaſon, the parallelogram EFGH is equal to the ſame EBCH: Therefore alſo the parallelogram ABCD is equal to EFGH. Wherefore parallelograms, &c. Q. E. D.

PROP. XXXVII. THEOR.

TRiangles upon the ſame baſe, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DBC be upon the ſame baſe BC and between the ſame parallels AD, BC: The triangle ABC is equal to the triangle DBC.



Produce AD both ways to the points E, F, and through B draw [a] BE parallel to CA; and through C draw CF parallel to BD: Therefore, each of the figures EBCA, DBCF is a parallelogram; and EBCA is equal [b] to DBCF, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, EF; and the triangle ABC is the half of the parallelogram EBCA, becauſe the diameter AB biſects [c] it; and the triangle DBC is the half of the parallelogram DBCF, becauſe the diameter DC biſects it: But the halves of equal things are equal [d]; therefore the triangle ABC is equal to the triangle DBC. Wherefore triangles, &c. Q. E. D.

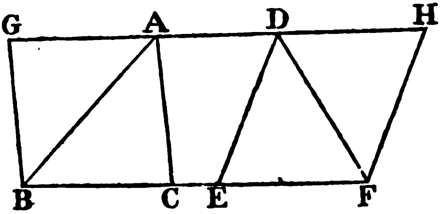
a 31. 1.
b 35. 1.
c 34. 1.
d 7. Ax.

PROP. XXXVIII. THEOR.

TRiangles upon equal baſes, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal baſes BC, EF, and between the ſame parallels BF, AD: The triangle ABC is equal to the triangle DEF.

Produce



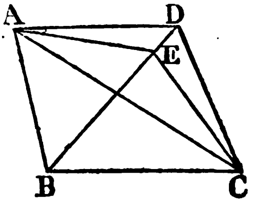
Produce AD both ways to the points G, H, and through B draw BG parallel [a] to CA, and through F draw FH parallel to ED: Then each of the figures GBCA, DEFH is a parallelogram; and they are equal to [b] one another, becauſe they are upon equal baſes BC, EF, and between the ſame parallels BF, GH; and the triangle ABC is the half [c] of the parallelogram GBCA, becauſe the diameter AB biſects it; and the triangle DEF is the half [c] of the parallelogram DEFH, becauſe the diameter DF biſects it: But the halves of equal things are equal [d]; therefore the triangle ABC is equal to the triangle DEF. Wherefore triangles, &c. Q. E. D.

a 31. 1.
b 36. 1.
c 34. 1.
d 7. Ax.

PROP. XXXIX. THEOR.

EQUAL triangles upon the ſame baſe, and upon the ſame ſide of it, are between the ſame parallels.

Let the equal triangles ABC, DBC be upon the ſame baſe BC, and upon the ſame ſide of it; they are between the ſame parallels.



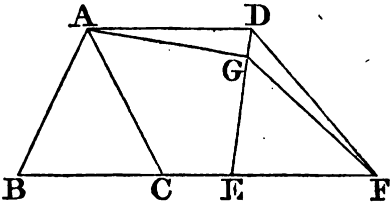
Join AD; AD is parallel to BC; for, if it is not, through the point A draw [a] AE parallel to BC, and join EC: The triangle ABC is equal [b] to the triangle EBC, becauſe it is upon the ſame baſe BC, and between the ſame parallels BC, AE: But the triangle ABC is equal to the triangle BDC; therefore alſo the triangle BDC is equal to the triangle EBC, the greater to the leſs, which is impoſſible: Therefore AE is not parallel to BC. In the ſame manner, it can be demonſtrated that no other line but AD is parallel to BC; AD is therefore parallel to it. Wherefore equal triangles upon, &c. Q. E. D.

a 31. 1.
b 37. 1.

PROP. XL. THEOR.

EQUAL triangles upon equal baſes, in the ſame ſtraight line, and towards the ſame parts, are between the ſame parallels.

Let the equal triangles ABC, DEF be upon equal baſes BC, EF, in the ſame ſtraight line BF, and towards the ſame parts; they are between the ſame parallels.



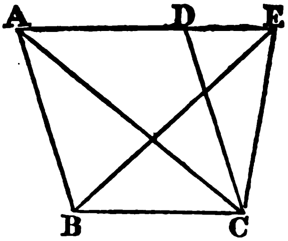
Join AD; AD is parallel to BC: For, if it is not, through A draw [a] AG parallel to BF, and join GF. The triangle ABC is equal [b] to the the triangle GEF, becauſe they are upon equal baſes BC, EF, and between the ſame parallels BF, AG: But the triangle ABC is equal to the triangle DEF; therefore alſo the triangle DEF is equal to the triangle GEF, the greater to the leſs, which is impoſſible: Therefore AG is not parallel to BF: And in the ſame manner it can be demonſtrated that there is no other parallel to it but AD; AD is therefore parallel to BF. Wherefore equal triangles, &c. Q. E. D.

a 31. 1.
b 38. 1.

PROP. XLI. THEOR.

IF a parallelogram and a triangle be upon the ſame baſe, and between the ſame parallels; the parallelogram ſhall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the ſame baſe BC, and between the ſame parallels BC, AE;



AE; the parallelogram ABCD is double of the triangle EBC.

Join AC; then the triangle ABC is equal [a] to the triangle EBC, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, AE. But the parallelogram ABCD is double [b] of the triangle ABC, becauſe the diameter AC divides it into two equal parts; wherefore ABCD is alſo double of the triangle EBC. Therefore, if a parallelogram, &c. Q. E. D.

a 37. 1.

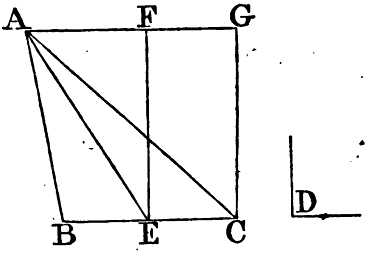
b 34. 1.

PROP. XLII. PROB.

TO deſcribe a parallelogram that ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle. It is required to deſcribe a parallelogram that ſhall be equal to the given triangle ABC, and have one of its angles equal to D.

Biſect [a] BC in E, join AE, and at the point E in the ſtraight line EC make [b] the angle CEF equal to D; and through A draw [c] AG parallel to EC, and through C draw CG [c] parallel to EF: Therefore FECG is a parallelogram: And becauſe BE is equal to EC, the triangle ABE is likewiſe equal [d] to the triangle AEC, ſince they are upon equal baſes BE, EC, and between the ſame parallels BC, AG; therefore the triangle ABC is double of the triangle AEC. And the parallelogram FECG is likewiſe double [e] of the triangle AEC, becauſe it is upon the ſame baſe, and between the ſame parallels: Therefore the parallelogram FECG is equal to the triangle ABC, and it has one of its angles CEF equal to the given angle D: Wherefore



a 10. 1.

b 23. 1.

c 31. 1.

d 38. 1.

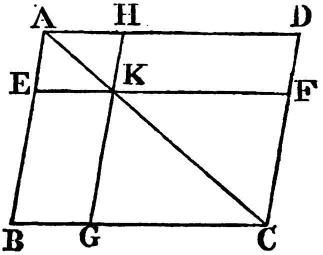
e 41. 1.

Wherefore there has been deſcribed a parallelogram FECG equal to a given triangle ABC, having one of it angles CEF equal to the given angle D. Which was to be done.

PROP. XLIII. THEOR.

THE complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diameter is AC; let EH, FG the parallelograms about AC, that is, thro' which AC paſſes, and BK, KD the other parallelograms which make up the whole figure ABCD, which are therefore called the complements: The complement BK is equal to the complement KD.



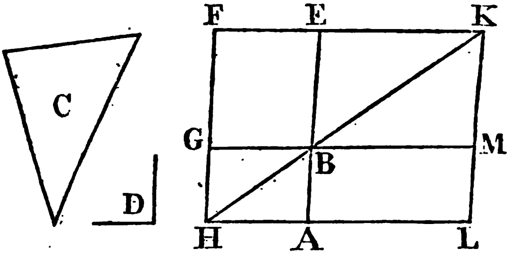
Becauſe ABCD is a parallelogram, and AC its diameter, the triangle ABC is equal[a] to the triangle ADC: And, becauſe EKHA is a parallelogram, the diameter of which is AK, the triangle AEK is equal to the triangle AHK: For the ſame reaſon, the triangle KGC is equal to the triangle KFC. Then, becauſe the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK, together with the triangle KGC is equal to the triangle AHK together with the triangle KFC: But the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal to the remaining complement KD. Wherefore the complements, &c. Q. E. D.

[a] 34. 1.

PROP. XLIV. PROB.

TO a given ſtraight line to apply a parallelogram, which ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given ſtraight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the ſtraight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



Make [a] the parallelogram BEFG equal to the triangle C, a 42. 1. and having the angle EBG equal to the angle D, ſo that BE be in the ſame ſtraight line with AB, and produce FG to H; and through A draw [b] AH parallel to BG or EF, and join b 31. 1. HB. Then becauſe the ſtraight line HF falls upon the parallels AH, EF, the angles AHF, HFE, are together equal [c] c 29. 1. to two right angles; wherefore the angles BHF, HFE are leſs than two right angles: But ſtraight lines which with another ſtraight line make the interior angles upon the ſame ſide leſs than two right angles, do meet if produced far enough: Therefore HB, FE ſhall meet, if produced; let them meet in K, and through K draw KL parallel to EA or FH, and produce HA, GB to the points L, M: Then HLKF is a parallelogram, of which the diameter is HK, and AG, ME are the parallelograms about HK; and LB, BF are the complements; therefore LB is equal [d] to BF: But BF is e- d 43. 1. qual to the triangle C; wherefore LB is equal to the triangle C; and becauſe the angle GBE is equal [e] to the angle e 15. 1. ABM, and likewiſe to the angle D; the angle ABM is equal to the angle D: Therefore the parallelogram

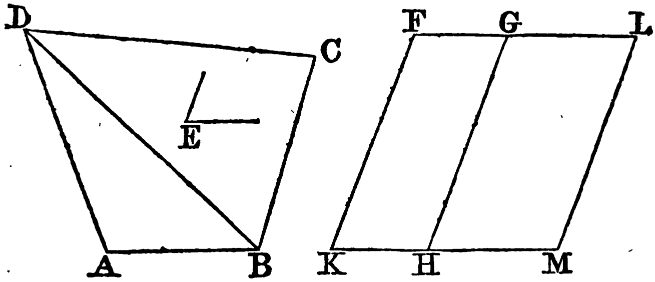
Book I. LB is applied to the ſtraight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D: Which was to be done.

PROP. XLV. PROB.

To deſcribe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to deſcribe a parallelogram equal to ABCD, and having an angle equal to E.

a 42. 1. Join DB, and deſcribe [a] the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle b 44. 1. E; and to the ſtraight line GH [b] apply the parallelogram GM equal to the triangle DBC, having the angle GHM equal to the angle E. And becauſe the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM; add to each of theſe the angle KHG; therefore the angles FKH, KHG are equal to the angles KHG, GHM; c 29. 1. but FKH, KHG are equal [c] to two right angles; therefore alſo KHG, GHM are equal to two right angles: and becauſe at the point H in the ſtraight line GH, the two ſtraight lines KH, HM, upon the oppoſite ſides of GH, make the adjacent angles equal to two right angles, KH is in the ſame ſtraight d 14. 1. line [d] with HM. And becauſe the ſtraight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal [c]; add to each of theſe the angle HGL; therefore the angles MHG, HGL, are equal to the angles HGF, HGL: But



But the angles MHG, HGL are equal[c] to two right angles; wherefore alſo the angles HGF, HGL are equal to two right angles, and FG is therefore in the ſame ſtraight line with GL. And becauſe KF is parallel to HG, and HG to ML, KF is parallel[e] to ML; but KM, FL are parallels; wherefore KFLM is a parallelogram. e 30. 1. And becauſe the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM, the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM; therefore the parallelogram KFLM has been deſcribed equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E. Which was to be done.

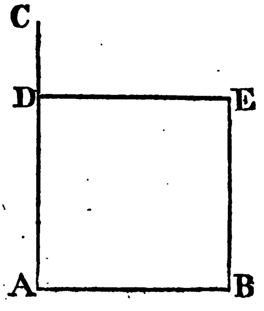
Cor. From this it is manifeſt how to a given ſtraight line to apply a parallelogram, which ſhall have an angle equal to a given rectilineal angle, and ſhall be equal to a given rectilineal figure, *viz.* by applying[b] to the given ſtraight line a parallelogram equal to the firſt triangle ABD, and having an angle equal to the given angle. b 44. 1.

PROP. XLVI. PROB.

To deſcribe a ſquare upon a given ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe a ſquare upon AB.

From the point A draw[a] AC at right angles to AB; and make[b] AD equal to AB, and through the point D draw DE parallel[c] to AB, and through B draw BE parallel to AD; therefore ADEB is a parallelogram: Whence AB is equal[d] to DE, and AD to BE: But BA is equal to AD; therefore the four ſtraight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral: it is likewiſe rectangular; for the ſtraight line AD meeting the parallels AB, DE, makes the angles BAD, ADE equal[e] to two right angles; but BAD is a right angle; therefore alſo ADE is a right angle; now the oppoſite angles of parallelo- a 11. 1. b 3. 1. c 31. 1. d 34. 1. e 29. 1.



grams are equal[d]; therefore each of the oppoſite an ABE, BED is a right angle; wherefore the figure AD is rectangular, and it has been demonſtrated that it is equi teral; it is therefore a ſquare, and it is deſcribed upon th ven ſtraight line AB: Which was to be done.

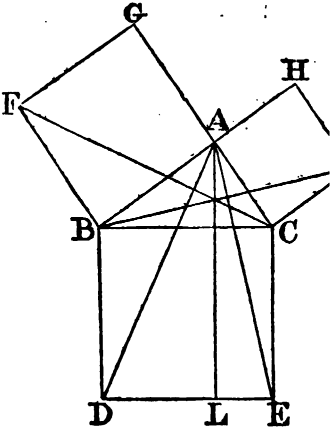
Cor. Hence every parallelogram that has one right a has all its angles right angles.

PROP. XLVII. THEOR.

IN any right angled triangle, the ſquare which deſcribed upon the ſide ſubtending the right gle, is equal to the ſquares deſcribed upon the ſi which contain the right angle.

Let ABC be a right angled triangle having the right an BAC; the ſquare deſcribed upon the ſide BC is equal to ſquares deſcribed upon BA, AC.

a 46. 1. On BC deſcribe[a] the ſquare BDEC, and on BA, AC
b 31. 1. ſquares GB, HC; and through A draw[b] AL parallel to or CE, and join AD, FC; then, becauſe each of the an BAC, BAG is a right
c 30. def. angle[c], the two ſtraight lines AC, AG upon the oppoſite ſides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the
d 14. 1. ſame ſtraight line[d] with AG; for the ſame reaſon, AB and AH are in the ſame ſtraight line; and becauſe the angle DBC is equal to the angle FBA, each of them being a right angle, add to each
e 2. Ax. the angle ABC, and the whole angle DBA is equal[e] to whole FBC; and becauſe the two ſides AB, BD are equa the two FB, BC, each to each, and the angle DBA equa



the angle FBC; therefore the baſe AD is equal[f] to the baſe FC, and the triangle ABD to the triangle FBC: Now the parallelogram BL is double[g] of the triangle ABD, becauſe they are upon the ſame baſe BD, and between the ſame parallels, BD, AL; and the ſquare GB is double of the triangle FBC, becauſe theſe alſo are upon the ſame baſe FB, and between the ſame parallels FB, GC. But the doubles of equals are equal[h] to one another; therefore the parallelogram BL is equal to the ſquare GB: And, in the ſame manner, by joining AE, BK, it is demonſtrated that the parallelogram CL is equal to the ſquare HC. Therefore the whole ſquare BDEC is equal to the two ſquares GB, HC; and the ſquare BDEC is deſcribed upon the ſtraight line BC, and the ſquares GB, HC upon BA, AC: Wherefore the ſquare upon the ſide BC is equal to the ſquares upon the ſides BA, AC. Therefore, in any right angled triangle, &c. Q. E. D.

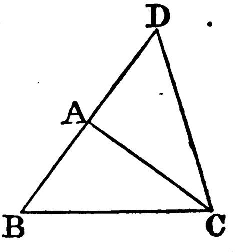
f 4. 1.
g 41. 1.
h 6 Ax.

PROP. XLVIII. THEOR.

IF the ſquare deſcribed upon one of the ſides of a triangle, be equal to the ſquares deſcribed upon the other two ſides of it; the angle contained by theſe two ſides is a right angle.

If the ſquare deſcribed upon BC, one of the ſides of the triangle ABC, be equal to the ſquares upon the other ſides BA, AC, the angle BAC is a right angle.

From the point A draw[a] AD at right angles to AC, and make AD equal to BA, and join DC: Then, becauſe DA is equal to AB, the ſquare of DA is equal to the ſquare of AB: To each of theſe add the ſquare of AC; therefore the ſquares of DA, AC, are equal to the ſquares of BA, AC: But the ſquare of DC is equal[b] to the ſquares of DA, AC, becauſe DAC is a right angle; and the ſquare of BC, by hypotheſis, is equal to the ſquares of BA, AC; therefore, the ſquare of DC is equal to the ſquare of BC; and therefore



a 11. 1.
b 47. 1.

therefore alſo the ſide DC is equal to the ſide BC. And becauſe the ſide DA is equal to AB, and AC common to the two triangles DAC, BAC, and the baſe DC likewiſe equal to the baſe BC, the angle DAC is equal c to the angle BAC: But DAC is a right angle; therefore alſo BAC is a right angle. Therefore, if the ſquare, &c. Q. E. D.

c 8. 1.

ELEMENTS

OF

GEOMETRY.

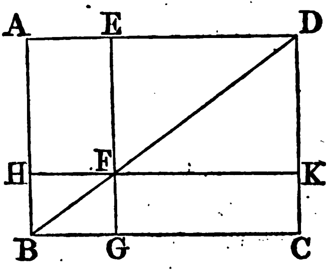
BOOK II.

DEFINITIONS.

I.

EVERY right angled parallelogram, *or rectangle*, is ſaid to be contained by any two of the ſtraight lines which contain one of the right angles.

II.

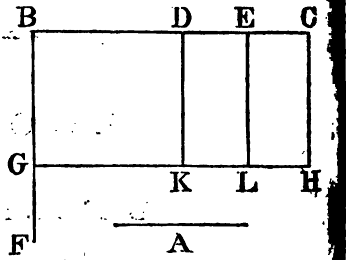


In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. 'Thus 'the parallelogram HG, 'together with the complements AF, FC, is 'the gnomon, which is 'more briefly expreſſed 'by the letters AGK, or 'EHC, which are at the 'oppoſite angles of the parallelograms which make the 'gnomon.'

PROP. I. THEOR.

IF there be two ſtraight lines, one of which is divided into any number of parts; the rectangle contained by the two ſtraight lines, is equal to the rectangles contained by the undivided line, and the ſeveral parts of the divided line.

Let A and BC be two ſtraight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the ſtraight lines A, BC is equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.

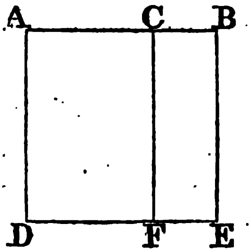


a 11. 1. From the point B draw [a] BF at right angles to BC, and
b 3. 1. make BG equal [b] to A; and
c 31. 1. through G draw [c] GH parallel to BC; and through D, E, C, draw [c] DK, EL, CH parallel to BG; then the rectangle BH is equal to the rectangles BK, DL, EH; and BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE,
d 34. 1. becauſe DK, that is, [d] BG, is equal to A; and in like manner the rectangle EH is contained by A, EC: Therefore the rectangle contained by A, BC is equal to the ſeveral rectangles contained by A, BD, and by A, DE, and alſo by A, EC. Therefore, if there be two ſtraight lines, &c. Q. E. D.

PROP. II. THEOR.

IF a ſtraight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the ſquare of the whole line.

Let the ſtraight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC, together with the rectangle * AB, AC, ſhall be equal to the ſquare of AB.



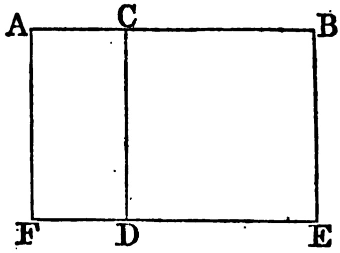
Upon AB deſcribe [a] the ſquare ADEB, and through C draw [b] CF, parallel to AD or BE: then AE is equal to the rectangles AF, CE; and AE is the ſquare of AB; and AF is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB: therefore the rectangle contained by AB, AC together with the rectangle AB, BC, is equal to the ſquare of AB. If therefore a ſtraight line, &c. Q. E. D.

a 46. 1.
b 31. 1.

PROP. III. THEOR.

If a ſtraight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the ſquare of the foreſaid part.

Let the ſtraight line AB be divided into two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB, together with the ſquare of BC.



Upon BC deſcribe [a] the ſquare CDEB, and produce ED to F, and through A draw [b] AF parallel to CD or BE; then the rectangle AE is equal to the rectangles AD, CE. Now AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is

a 46. 1.
b 31. 1.

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* N. B. To avoid repeating the word *contained* too frequently, the rectangle contained by two ſtraight lines AB, AC is ſometimes ſimply called the rectangle AB, AC.

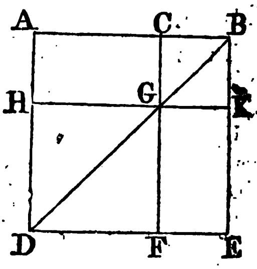
Book II equal to CB; and DB is the ſquare of BC: therefore the rectangle AB BC is equal to the rectangle AC, CB together with the ſquare of BC. If therefore a ſtraight, &c. Q. E. D.

PROP. IV. THEOR.

IF a ſtraight line be divided into any two parts, the ſquare of the whole line is equal to the ſquares of the two parts, together with twice the rectangle contained by the parts.

Let the ſtraight line AB be divided into any two parts in C; the ſquare of AB is equal to the ſquares of AC, CB, and to twice the rectangle contained by AC, CB.

a 46. 1. Upon AB deſcribe [a] the ſquare ADEB, and join BD, and
b 31. 1 through C draw [b] CGF parallel to AD or BE, and through
G draw HK parallel to AB or DE. And becauſe CF is parallel
to AD, and BD falls upon them, the exterior angle
c 29. 1. BGC is equal [c] to the interior and oppoſite angle ADB; but
d 5 1. ADB is equal [d] to the angle ABD, becauſe BA is equal to
AD, being ſides of a ſquare; wherefore the angle CGB is
e 6. 1. equal to the angle GBC; and therefore the ſide BC is equal [e]
f 34. 1. to the ſide CG: but CB is equal [f]
alſo to GK, and CG to BK; wherefore
the figure CGKB is equilateral.
It is likewiſe rectangular; for
the angle CBK being a right angle,
the other angles of the parallelogram
g Cor. 46. 1. CGKB are alſo right angles [g].
Wherefore CGKB is a ſquare, and
it is upon the ſide CB. For the
ſame reaſon HF alſo is a ſquare,
and it is upon the ſide HG, which is equal to AC; therefore
HF, CK are the ſquares of AC, CB. And becauſe the
h 43. 1. complement AG is equal [h] to the complement GE, and
becauſe AG is the rectangle contained by AC, CG, that
is, by AC, CB; GE is alſo equal to the rectangle AC, CB;
wherefore AG, GE are equal to twice the rectangle AC,
CB. And HF, CK are the ſquares of AC, CB; wherefore the
four figures HF, CK, AG, GE are equal to the ſquares of



AC, CB, and to twice the rectangle AC, CB: But HF, CK, AG, GE make up the whole figure ADEB, which is the ſquare of AB: Therefore the ſquare of AB is equal to the ſquares of AC, CB, and twice the rectangle AC, CB. Wherefore if a ſtraight line, &c. Q. E. D.

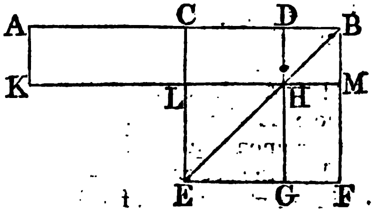
Cor. From the demonſtration, it is manifeſt that the parallelograms about the diameter of a ſquare are likewiſe ſquares.

PROP. V. THEOR.

IF a ſtraight line be divided into two equal parts, and alſo into two unequal parts; the rectangle contained by the unequal parts, together with the ſquare of the line between the points of ſection, is equal to the ſquare of half the line.

Let the ſtraight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB, together with the ſquare of CD, is equal to the ſquare of CB.

Upon CB deſcribe [a] the ſquare CEFB, join BE, and — a 46. 1.
through D draw [b] DHG parallel to CE or BF; and through — b 31. 1.
H draw KLM parallel to CB or EF; and alſo through A draw AK parallel to CL or BM: And becauſe the complement CH is equal [c] to the complement HF, to each of theſe — c 43. 1.
add DM; therefore the whole CM is equal to the whole DF; but CM is equal [d] to AL, — d 36. 1.
becauſe AC is equal to CB; therefore alſo AL is equal to DF. To each of theſe add CH, and the whole AH is equal to DF and CH: but AH is the rectangle contained by AD, DB, for DH is equal [e] to — e Cor 4. 2.
DB; and DF together with CH is the gnomon CMG; therefore the gnomon CMG is equal to the rectangle AD, DB. To each of theſe add LG, which is equal [e] to the ſquare of CD; therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the



)

ſquare of CD: But the gnomon CMG and LG make up the whole figure CEFB, which is the ſquare of CB: Therefore the rectangle AD, DB, together with the ſquare of CD, is equal to the ſquare of CB. Wherefore, if a ſtraight line, &c. Q. E. D.

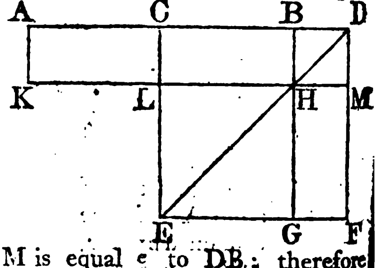
From this propoſition it is manifeſt, that the difference of the ſquares of two unequal lines AC, CD, is equal to the rectangle contained by their ſum and difference.

PROP. VI. THEOR.

IF a ſtraight line be biſected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the ſquare of half the line biſected, is equal to the ſquare of the ſtraight line which is made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the rectangle AD, DB, together with the ſquare of CB, is equal to the ſquare of CD.

a 46. 1. Upon CD deſcribe [a] the ſquare CEFD, join DE, and
b 31. 1. through B draw [b] BHG parallel to CE or DF, and through H draw KLM parallel to AD or EF, and alſo through A draw AK parallel to CL or DM: and becauſe AC is equal to CB, the rectangle AL
c 36. 1. is equal [c] to CH; but
d 43. 1. CH is equal [d] to HF; therefore alſo AL is equal to HF: To each of theſe add CM; therefore the whole AM is equal to the gnomon CMG. Now AM is the rectangle con-



e Cor. 4. 2. tained by AD, DB, for DM is equal [e] to DB: therefore the gnomon CMG is equal to the rectangle AD, DB; add to each of theſe LG, which is equal to the ſquare of CB, therefore the rectangle AD, DB, together with the ſquare of CB, is equal to the gnomon CMG, together with LG: But the gnomon CMG together with LG makes up the whole figure

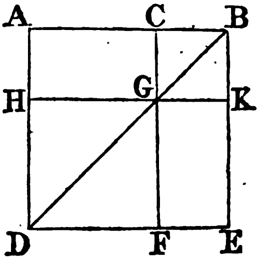
CEFD, which is the ſquare of CD; therefore the rectangle AD, DB, together with the ſquare of CB, is equal to the ſquare of CD. Wherefore, if a ſtraight line, &c. Q. E. D.

PROP. VII. THEOR.

IF a ſtraight line be divided into any two parts, the ſquares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the ſquare of the other part.

Let the ſtraight line AB be divided into any two parts in the point C; the ſquares of AB, BC are equal to twice the rectangle AB, BC together with the ſquare of AC.

Upon AB deſcribe [a] the ſquare ADEB, and conſtruct the figure as in the preceding propoſitions: And becauſe AG is equal [b] to GE, add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK, CE, are double of AK: But AK, CE are the gnomon AKF together with the ſquare CK; therefore the gnomon AKF, together with the ſquare CK, is double of AK. But twice the rectangle AB, BC is alſo double of AK, for BK is equal [c] to BC: therefore the gnomon AKF, together with the ſquare CK, is equal to twice the rectangle AB, BC. To each of theſe equals add HF, which is equal to the ſquare of AC; therefore the gnomon AKF, together with the ſquares CK, HF, is equal to twice the rectangle AB, BC and the ſquare of AC: now the gnomon AKF, together with the ſquares CK, HF, makes up the whole figure ADEB and CK, which are the ſquares of AB and BC: Therefore the ſquares of AB and BC are equal to twice the rectangle AB, BC, together with the ſquare of AC. Wherefore, if a ſtraight line, &c. Q. E. D.



[a] 46. 1.

[b] 43. 1.

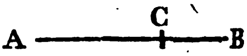
[c] Cor. 4. 2.

Otherwiſe,

“ Becauſe the ſquare of AB is equal to the ſquares of AC “ and CB, [a] together with twice “ the rectangle ACB, if the “ ſquare of CB be added to both, “ the ſquares of AB and CB are equal to the ſquare of “ AC together with twice the ſquare of CB and twice the “ rectangle ACB. But the ſquare of CB together with the “ rectangle ACB is equal to the [b] rectangle ABC; and “ therefore twice the ſquare of CB, together with twice the “ rectangle ACB, is equal to twice the rectangle ABC. Where- “ fore the ſquares of AB and CB are equal to twice the rect- “ angle ACB, together with the ſquare of AC. Q. E. D.

a 4. 2.

b 3. 2.



“ Cor. Hence, the ſum of the ſquares of any two lines is “ equal to twice the rectangle contained by the lines together “ with the ſquare of the difference of the lines.”

PROP. VIII. THEOR.

IF a ſtraight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the ſquare of the other part, is equal to the ſquare of the ſtraight line which is made up of the whole and the firſt mentioned part.

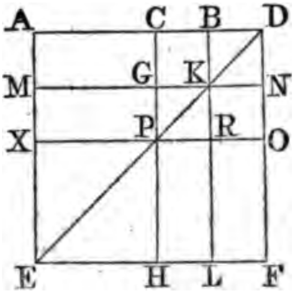
Let the ſtraight line AB be divided into any two parts in the point C; four times the rectangle AB, BC, together with the ſquare of AC, is equal to the ſquare of the ſtraight line made up of AB and BC together.

Produce AB to D, ſo that BD be equal to CB, and upon AD deſcribe the ſquare AEFD; and conſtruct two figures ſuch as in the preceding. Becauſe CB is equal to BD, and that CB is equal [a] to GK, and BD to KN; therefore GK is equal to KN: For the ſame reaſon, PR is equal to RO; and becauſe CB is equal to BD, and GK to KN, the rectangle CK is equal [b] to BN, and GR to RN: But CK is equal [c] to RN, becauſe they are the complements of the parallelogram CO; therefore alſo BN is equal to GR; and the four rectangles BN, CK, GR, RN are therefore equal to one another, and

a 34. 1.

b 36. 1.

c 43. 1.



and ſo are quadruple of one of them, CK. Again, becauſe CB is equal to BD, and BD equal [d] to BK, that is, to CG; and CB equal to GK, that [d] is, to GP; therefore CG is equal to GP; and becauſe CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF: But MP is equal [e] to PL, becauſe they are the complements of the parallelogram ML; wherefore AG is equal alſo to RF: Therefore the four rectangles AG, MP, PL, RF are equal to one another, and ſo are quadruple of one of them, AG. And it was demonſtrated, that the four CK, BN, GR, and RN are quadruple of CK: therefore the eight rectangles which make up the gnomon AOH, are quadruple of AK. And becauſe AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB, BC, is quadruple of AK: But the gnomon AOH was demonſtrated to be quadruple of AK; therefore four times the rectangle AB, BC, is equal to the gnomon AOH. To each of theſe add XH, which is equal [d] to the ſquare of AC; therefore four times the rectangle AB, BC together with the ſquare of AC, is equal to the gnomon AOH and the ſquare XH: But the gnomon AOH and the ſquare XH make up the figure AEFD, which is the ſquare of AD: Therefore four times the rectangle AB, BC, together with the ſquare of AC, is equal to the ſquare of AD, that is, of AB and BC added together in one ſtraight line. Wherefore, if a ſtraight line, &c. Q. E. D.

d Cor. 4. 2.

e 43. 1.

d Cor. 4. 2.

" Cor. 1. Hence, becauſe AD is the ſum, and AC the dif-
" ference of the lines AB and BC, four times the rectangle
" contained by any two lines together with the ſquare of
" their difference, is equal to the ſquare of the ſum of the
" lines."

" Cor. 2. From the demonſtration it is manifeſt, that ſince
" the ſquare of CD is quadruple of the ſquare of CB, the
" ſquare of any line is quadruple of the ſquare of half that
" line."

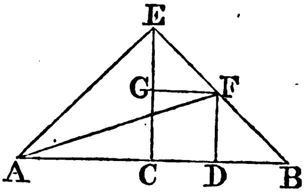
PROP.

PROP. IX. THEOR.

IF a ſtraight line be divided into two equal, and alſo into two unequal parts; the ſquares of the two unequal parts are together double of the ſquare of half the line, and of the ſquare of the line between the points of ſection.

Let the ſtraight line AB be divided at the point C into two equal, and at D into two unequal parts: The ſquares of AD, DB are together double of the ſquares of AC, CD.

a 11. 1. From the point C draw [a] CE at right angles to AB, and
make it equal to AC or CB, and join EA, EB; through D
b 31. 1. draw [b] DF parallel to CE, and through F draw FG parallel
to AB; and join AF: Then, becauſe AC is equal to CE, the
c. 5. 1. angle EAC is equal [c] to the angle AEC; and becauſe the
angle ACE is a right angle, the two others AEC, EAC together
d. 32. 1. make one right angle [d]; and they are equal to one another;
each of them therefore is half of a right angle. For the ſame reaſon each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle: And becauſe the angle GEF is half a right angle, and EGF a



e. 29. 1. right angle, for it is equal [e] to the interior and oppoſite angle
ECB, the remaining angle EFG is half a right angle; therefore
the angle GEF is equal to the angle EFG, and the ſide
f 6. 1. EG equal [f] to the ſide GF: Again, becauſe the angle at B is
half a right angle and FDB a right angle, for it is equal [e] to the interior and oppoſite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the ſide DF to [f] the ſide DB. Now becauſe AC is equal to CE, the ſquare of AC is equal to the ſquare of CE; therefore the ſquares of AC, CE, are double
g 47. 1. of the ſquare of AC: But the ſquare of EA is equal [g] to the
ſquares of AC, CE, becauſe ACE is a right angle; therefore the ſquare of EA is double of the ſquare of AC: Again, becauſe EG is equal to GF, the ſquare of EG is equal to the ſquare of GF; therefore the ſquares of EG, GF are double of the

the ſquare of GF; but the ſquare of EF is equal to the ſquares of EG, GF; therefore the ſquare of EF is double of the ſquare GF; and GF is equal [h] to CD; therefore the ſquare of EF is double of the ſquare of CD: But the ſquare of AE is likewiſe double of the ſquare of AC; therefore the ſquares of AE, EF are double of the ſquares of AC, CD: And the ſquare of AF is equal [i] to the ſquares of AE, EF, becauſe AEF is a right angle; therefore the ſquare of AF is double of the ſquares of AC, CD: But the ſquares of AD, DF are equal to the ſquare of AF, becauſe the angle ADF is a right angle; therefore the ſquares of AD, DF are double of the ſquares of AC, CD: And DF is equal to DB; therefore the ſquares of AD, DB are double of the ſquares of AC, CD. If therefore a ſtraight line, &c. Q. E. D.

h 34. 1.
i 47. 1.

PROP. X. THEOR.

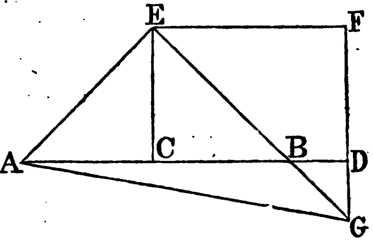
IF a ſtraight line be biſected, and produced to any point, the ſquare of the whole line thus produced, and the ſquare of the part of it produced, are together double of the ſquare of half the line biſected, and of the ſquare of the line made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the ſquares of AD, DB are double of the ſquares of AC, CD.

From the point C draw [a] CE at right angles to AB: And make it equal to AC or CB, and join AE, EB; through E draw [b] EF parallel to AB, and through D draw DF parallel to CE: And becauſe the ſtraight line EF meets the parallels EC, FD, the angles CEF, EFD are equal [c] to two right angles; and therefore the angles BEF, EFD are leſs than two right angles: But ſtraight lines, which with another ſtraight line make the interior angles upon the ſame ſide leſs than two right angles, do meet [d] if produced far enough: Therefore EB, FD ſhall meet, if produced towards B, D; let them meet in G, and join AG: Then, becauſe AC is equal to CE, the angle CEA is equal [e] to the angle EAC; and the angle ACE is a right angle; therefore each of the angles CEA, EAC is half

a 11. 1.
b 31. 1.
c 29. 1.
d Cor. 29. 1.
e 5. 1.

half a right angle[f]: For the ſame reaſon, each of the angles
f 32. 1. CEB, EBC is half a right angle; therefore AEB is a right
angle: And becauſe EBC is half a right angle, DBG is alſo
g 15. 1. [g] half a right angle, for they are vertically oppoſite; but
c 29. 1. BDG is a right angle, becauſe it is equal[c] to the alternate
angle DCE; therefore the remaining angle DGB is half a
right angle, and is therefore equal to the angle DBG; where-
h 6. 1. fore alſo the ſide DB is equal[h] to the ſide DG: Again, be-
cauſe EGF is half a right
angle, and that the angle at
F is a right angle, becauſe
i 34. 1. it is equal[i] to the oppoſite
angle ECD, the remaining
angle FEG is half a right
angle, and equal to the
angle EGF; wherefore alſo
the ſide GF is equal[h] to the
ſide FE. And becauſe EC is equal to CA, the ſquare of EC
is equal to the ſquare of CA; therefore the ſquares of EC,
CA are double of the ſquare of CA: But the ſquare of EA
k 47. 1. is equal[k] to the ſquares of EC, CA; therefore the ſquare of
EA is double of the ſquare of AC: Again, becauſe GF is
equal to FE, the ſquare of GF is equal to the ſquare of FE;
and therefore the ſquares of GF, FE are double of the ſquare
of EF: But the ſquare of EG is equal[k] to the ſquares of GF,
FE; therefore the ſquare of EG is double of the ſquare of
EF; and EF is equal to CD; wherefore the ſquare of EG is
double of the ſquare of CD; but it was demonſtrated, that
the ſquare of EA is double of the ſquare of AC; therefore
the ſquares of AE, EG are double of the ſquares of AC, CD:
And the ſquare of AG is equal[k] to the ſquares of AE, EG;
therefore the ſquare of AG is double of the ſquares of AC,
CD: But the ſquares of AD, DG are equal[k] to the ſquare of
AG; therefore the ſquares of AD, DG are double of the
ſquares of AC, CD: But DG is equal to DB; therefore the
ſquares of AD, DB are double of the ſquares of AC, CD:
Wherefore, if a ſtraight line, &c. Q. E. D.



PROP.

PROP. XI. PROB.

TO divide a given ſtraight line into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

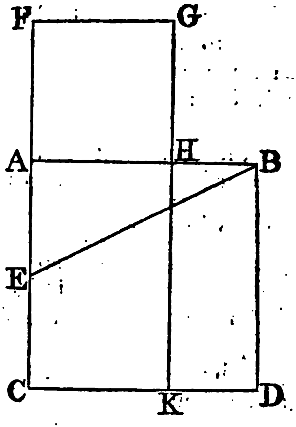
Let AB be the given ſtraight line; it is required to divide it into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

Upon AB deſcribe [a] the ſquare ABDC; biſect [b] AC in E, and join BE; produce CA to F, and make [c] EF equal to EB, and upon AF deſcribe [a] the ſquare FGHA; AB is divided in H, ſo that the rectangle AB, BH is equal to the ſquare of AH.

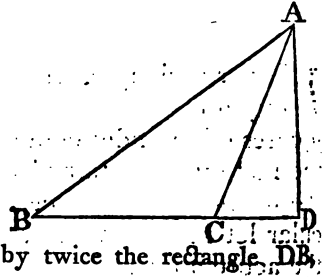
a 46. 1.
b 10. 1.
c 3. 1.

Produce GH to K: Becauſe the ſtraight line AC is biſected in E, and produced to the point F, the rectangle CF, FA, together with the ſquare of AE, is equal [d] to the ſquare of EF: But EF is equal to EB; therefore the rectangle CF, FA, together with the ſquare of AE, is equal to the ſquare of EB: And the ſquares of BA, AE are equal [e] to the ſquare of EB, becauſe the angle EAB is a right angle; therefore the rectangle CF, FA, together with the ſquare of AE, is equal to the ſquares of BA, AE: take away the ſquare of AE, which is common to both, therefore the remaining rectangle CF, FA is equal to the ſquare of AB. Now the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the ſquare of AB; therefore FK is equal to AD: take away the common part AK, and the remainder FH is equal to the remainder HD: And HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the ſquare of AH; therefore the rectangle AB, BH is

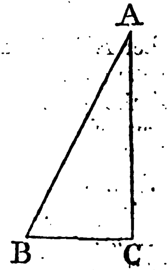
d 6. 2.
e 47. 1.



Book II. CD: But becauſe BD is divided into two parts in C, the rect-
f 3. 2. angle DB, BC is equal [f] to the rectangle BC, CD and the ſquare of BC: And the doubles of theſe are equal: Therefore the ſquares of AB, BC are equal to the ſquare of AC, and twice the rectangle DB, BC: Therefore the ſquare of AC alone is leſs than the ſquares of AB, BC by twice the rectangle DB, BC.



Laſtly, Let the ſide AC be perpendicular to BC; then is BC the ſtraight line between the perpendicular and the acute angle at B; and it is manifeſt that
g 47. 1. the ſquares of AB, BC are equal [g] to the ſquare of AC, and twice the ſquare of BC: Therefore, in every triangle, &c. Q. E. D.



PROP. XIV. PROB.

TO deſcribe a ſquare that ſhall be equal to a given rectilineal figure.

Let A be the given rectilineal figure; it is required to deſcribe a ſquare that ſhall be equal to A.

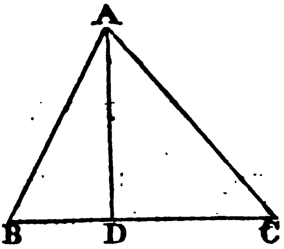
a 45. 1. Deſcribe [a] the rectangular parallelogram BCDE equal to the rectilineal figure A. If then the ſides of it BE, ED are equal to one another, it is a ſquare, and what was required is done; but if they are not equal, produce one of them BE to F, and make EF equal to ED, and biſect BF in G: and from the centre G, at the diſtance GB, or GF, deſcribe the ſemicircle BHF, and produce DE to H, and join GH. Therefore, becauſe the ſtraight line BF is divided into two equal parts in the point G, and into two unequal in the point E, the rectangle BE, EF, together with the ſquare of EG, is
b 5. 2. equal [b] to the ſquare of GF: but GF is equal to GH; therefore

PROP. XIII. THEOR.

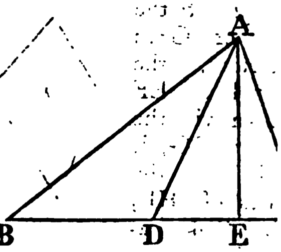
IN every triangle, the ſquare of the ſide ſubtending any of the acute angles, is leſs than the ſquares of the ſides containing that angle, by twice the rectangle contained by either of theſe ſides, and the ſtraight line intercepted between the perpendicular let fall upon it from the oppoſite angle, and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the ſides containing it, let fall the perpendicular [a] AD from the oppoſite angle: The ſquare of AC, oppoſite to the angle B, is leſs than the ſquares of CB, BA by twice the rectangle CB, BD. a 12. 1.

First, Let AD fall within the triangle ABC; and becauſe the ſtraight line CB is divided into two parts in the point D, the ſquares of CB, BD are equal [b] to twice the rectangle contained by CB, BD, and the ſquare of DC: To each of theſe equals add the ſquare of AD; therefore the ſquares of CB, BD, DA are equal to twice the rectangle CB, BD, and the ſquares of AD, DC: But the ſquare of AB is equal [c] to the ſquares of BD, DA, becauſe the angle BDA is a right angle; and the ſquare of AC is equal to the ſquares of AD, DC: Therefore the ſquares of CB, BA are equal to the ſquare of AC, and twice the rectangle CB, BD, that is, the ſquare of AC alone is leſs than the ſquares of CB, BA by twice the rectangle CB, BD. b 7. 2. c 47. 1.



Secondly, Let AD fall without the triangle ABC: Then, becauſe the angle at D is a right angle, the angle ACB is greater [d] than a right angle; and therefore the ſquare of AB is equal [e] to the ſquares of AC, CB, and twice the rectangle BC, CD: To theſe equals add the ſquare of BC, and the ſquares of AB, BC are equal to the ſquare of AC, and twice the ſquare of BC, and twice the rectangle BC, CD: d 16 1. e 12. 2.



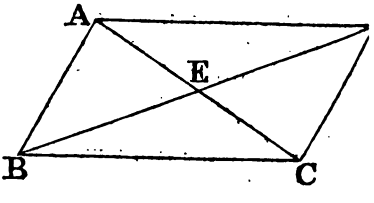
Book II. line BC is cut equally in D, and unequally in E, the ſqua
b 9. 2. of BE and EC [b] are equal to twice the ſquares of BD and DE; therefore the ſquares of BA and AC are equal to twice the ſquare of BD, together with twice the ſquares of DE and EA. Now, the ſquares of DE and EA are equal to the ſquare of DA [a], and therefore twice the ſquares of DE and EA, to twice the ſquare of DA. Wherefore alſo, the ſquares of BA and AC are equal to twice the ſquare of BD, toget
with twice the ſquare of DA. Therefore, &c. Q. E. D.

PROP. B. THEOR.

THE ſum of the ſquares of the diameters of a parallelogram is equal to the ſum of t
ſquares of the ſides of the parallelogram.

Let ABCD be a parallelogram, of which the diameters
AC and BD; the ſum of the ſquares of AC and BD is eq
to the ſum of the ſquares of AB, BC, CD, DA.

Let AC and BD interſect one another in E: and beca
a 15. 1. the vertical angles AED, CEB are equal [a], and alſo the
b 29. 1. ternate angles EAD, ECB [b], the triangles ADE, CEB h
two angles in the one equal to two angles in the other, e
to each: but the ſides AD and BC, which are oppoſite
equal angles in theſe triangles,
c 34. 1. are alſo equal [c]; therefore the other ſides which are oppoſite to the equal angles are
d 26. 1. equal [d], viz. AE to EC, and ED to EB.



Since, therefore, BD is biſected in E, the ſquares of BA and AD are equal to twice the ſquare of BE, together with twice the ſquar
e A. 2. EA [e]; and for the ſame reaſon the ſquare of BC and (

are equal to twice the ſquare of BE, together with twice the ſquare of EC, that is, of EA, becauſe EC is equal to EA. Therefore the four ſquares of BA, AD, DC, CB are equal to four times the ſquares of BE and of EA. But the ſquare of BD is equal to four times the ſquare of BE, becauſe BD is double of BE e; and for the ſame reaſon, the ſquare of AC is equal to four times the ſquare of AE: Wherefore alſo, the ſquares of BD and AC are equal to the four ſquares of BA, AD, DC, CB. Therefore the ſum of the ſquares, &c. Q. E. D.

e Cor. 8. 2.

Cor. From the demonſtration, it is manifeſt that the diameters of every parallelogram biſect one another.

ELEMENTS

OF

GEOMETRY.

BOOK III.

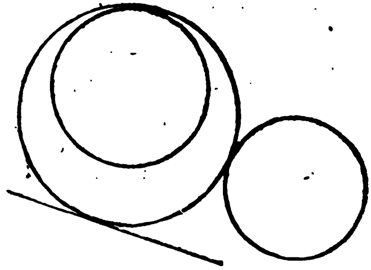
DEFINITIONS.

I.

A ſtraight line is ſaid to touch a circle, when it meets the circle, and being produced does not cut it.

II.

Circles are ſaid to touch one another, which meet,but do not cut one another.

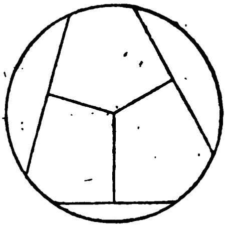


III.

Sraight lines are ſaid to be equally diſtant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

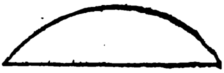
IV.

And the ſtraight line on which the greater perpendicular falls, is ſaid to be farther from the centre.



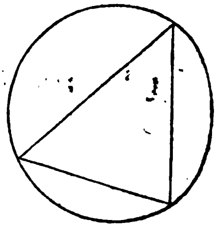
V.

A ſegment of a circle is the figure contained by a ſtraight line and the circumference it cuts off.



VI.

An angle in a ſegment is the angle contained by two ſtraight lines drawn from any point in the circumference of the ſegment, to the extremities of the ſtraight line which is the baſe of the ſegment.

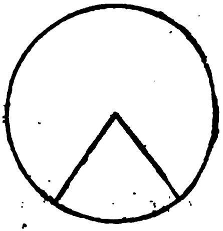


VII.

And an angle is ſaid to inſiſt or ſtand upon the circumference intercepted between the ſtraight lines that contain the angle.

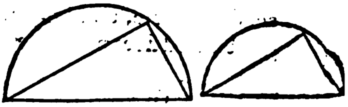
VIII.

The ſector of a circle is the figure contained by two ſtraight lines drawn from the centre, and the circumference between them.



XI.

Similar ſegments of a circle, are thoſe in which the angles are equal, or which contain equal angles.



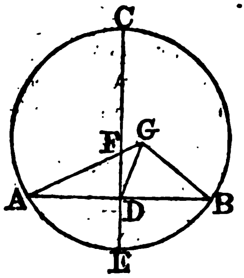
PROP. I. PROB.

TO find the centre of a given circle.

Let ABC be the given circle; it is required to find its centre.

Draw within it any ſtraight line AB, and biſect[a] it in D; from the point D draw[b] DC at right angles to AB, and produce it to E, and biſect CE in F: The point F is the centre of the circle ABC.

a 10. 1.
b 11. 1.



For, if it be not, let, if poſſible, G be the centre, and join GA, GD, GB: Then, becauſe DA is equal to DB, and DG common to the two triangles ADG, BDG, the two ſides AD, DG are equal to the two BD, DG, each to each; and the baſe GA is equal to the baſe GB, becauſe they are drawn from the centre G*: therefore the angle ADG is equal[c] to the angle GDB: But when a ſtraight line ſtanding upon another ſtraight line makes the adjacent angles equal to one another, each of the angles is a right angle[d]. Therefore the angle GDB is a right angle: But FDB is likewiſe a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the leſs, which is impoſſible: Therefore G is not the centre of the circle ABC: In the ſame manner it can be ſhown, that no other point but F is the centre; that is, F is the centre of the circle ABC: Which was to be found.

c 8. 1.
d 7 def. 1.

Cor. From this it is manifeſt, that if in a circle a ſtraight line biſect another at right angles, the centre of the circle is in the line which biſects the other.

PROP. II. THEOR.

IF any two points be taken in the circumference of a circle, the ſtraight line which joins them ſhall fall within the circle.

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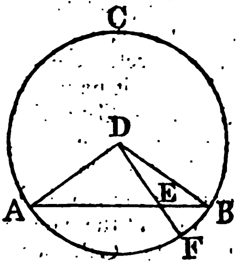
* N.B. Whenever the expreſſion "ſtraight lines from the centre," or "drawn from the centre," occurs, it is to be underſtood that they are drawn to the circumference.

Book III. Let ABC be a circle, and A, B any two points in the circumference; the ſtraight line drawn from A to B ſhall fall within the circle.



Take any point in AB as E; find D the centre of the circle ABC, join AD, DB and DE, and let DE, meet the circumference in F. Then becauſe DA is equal to DB, the angle DAB is equal[a] to the angle DBA; and becauſe AE, a ſide of the triangle DAE, is produced to B, the angle DEB is greater[b] than the angle DAE; but DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE: Now to the greater angle the greater ſide is oppoſite[d]; DB is therefore greater than DE: but DB is equal to DF; wherefore DF is greater than DE, and the point E is therefore within the circle. The ſame may be demonſtrated of any other point between A and B, therefore AB is within the circle. Wherefore, if any two points, &c. Q. E. D.

a 5. 1.
b 16. 1.
c 19. 1.

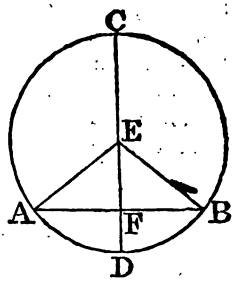
PROP. III. THEOR.

IF a ſtraight line drawn through the centre of a circle biſect a ſtraight line in it which does not paſs through the centre, it ſhall cut it at right angles; and, if it cuts it at right angles, it ſhall biſect it.

Let ABC be a circle, and let CD, a ſtraight line drawn through the centre, biſect any ſtraight line AB, which does not paſs through the centre, in the point F: It cuts it alſo at right angles.

a 1. 3. Take[a] E the centre of the circle, and join EA, EB. Then, becauſe AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two ſides in the one equal to two ſides in the other, and the baſe EA is equal to the baſe EB; therefore the angle AFE is equal [b] to the angle BFE: But when a ſtraight line ſtanding upon another makes the adjacent angles equal to one another, each of them is a right [c] angle: Therefore each of the angles AFE, BFE is a right angle; wherefore the ſtraight line CD, drawn through the centre biſecting another AB that does not paſs through the centre, cuts the ſame at right angles.

b 8. 1.
c 7. def. 1.



But let CD cut AB at right angles; CD alſo biſects it, that is, AF is equal to FB.

The ſame conſtruction being made, becauſe EA, EB from the centre are equal to one another, the angle EAF is equal [d] to the angle EBF; and the right angle AFE is equal to the right angle BFE: Therefore, in the two triangles EAF, EBF, there are two angles in one equal to two angles in the other, and the ſide EF, which is oppoſite to one of the equal angles in each, is common to both; therefore the other ſides are equal [e]; AF therefore is equal to FB. Wherefore, if a ſtraight line &c. Q. E. D.

d 5. 1.
e 26. 1.

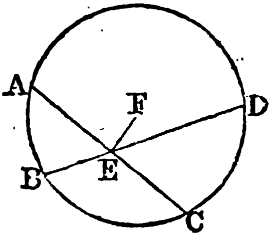
PROP. IV. THEOR.

IF in a circle two ſtraight lines cut one another which do not both paſs through the centre, they do not biſect each the other.

Let ABCD be a circle, and AC, BD two ſtraight lines in it, which cut one another in the point E, and do not both paſs through the centre: AC, BD do not biſect one another.

For, if it is poſſible, let AE be equal to EC, and BE to ED: If one of the lines paſs through the centre, it is plain that it cannot be biſected by the other which does not paſs through the centre. But if neither of them paſs through the centre, take [a] F the centre of the circle, and join EF: and becauſe FE, a ſtraight line through the centre, biſects another AC which does not paſs thro' the centre, it ſhall cut it at right [b] angles; wherefore FEA is a right

a 1. 3.
b



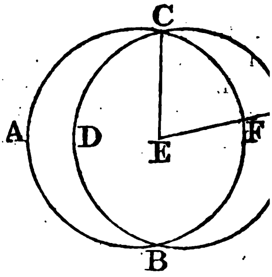
angle: Again, becauſe the ſtraight line FE biſects the ſtr line BD, which does not paſs through the centre, it ſhal it at right [b] angles; wherefore FEB is a right angle: FEA was ſhown to be a right angle; therefore FEA qual to the angle FEB, the leſs to the greater, which is poſſible: therefore AC, BD do not biſect one another. Wl fore, if in a circle, &c. Q. E. D.

PROP. V. THEOR.

IF two circles cut one another, they ſhall have the ſame centre.

Let the two circles ABC, CDG cut one another in points B, C; they have not the ſame centre.

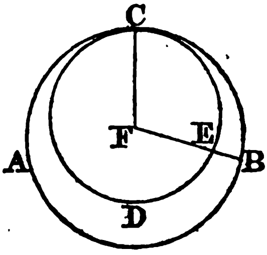
For, if it be poſſible, let E be their centre: join EC, draw any ſtraight line EFG meeting them in F and G: and becauſe E is the centre of the circle ABC, CE is equal to EF: again, becauſe E is the centre of the circle CDG CE is equal to EG: but, CE was ſhown to be equal to EF, therefore EF is equal to EG, the leſs to the greater, which is impoſſible: therefore E is not the centre of the circles ABC, CDG. Wherefore, if circles, &c. Q. E. D.



PROP. VI. THEOR.

IF two circles touch one another internally, t ſhall not have the ſame centre.

Let the two circles ABC, CDE, touch one another i nally in the point C: they have not the ſame centre.



For, if they can, let it be F; join FC, and draw any ſtraight line FEB meeting them in E and B; and becauſe F is the centre of the circle ABC, CF is equal to FB; alſo, becauſe F is the centre of the circle CDE, CF is equal to FE: and CF was ſhown equal to FB; therefore FE is equal to FB, the leſs to the greater, which is impoſſible; wherefore F is not the centre of the circles ABC, CDE. Therefore, if two circles, &c. Q. E. D.

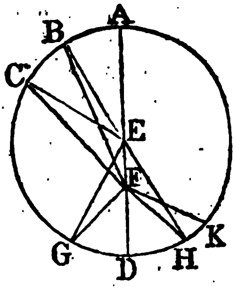
PROP. VII. THEOR.

IF any point be taken in the diameter of a circle which is not the centre, of all the ſtraight lines which can be drawn from it to the circumference, the greateſt is that in which the centre is, and the other part of that diameter is the leaſt; and, of any others, that which is nearer to the line which paſſes through the centre is always greater than one more remote: And from the ſame point there can be drawn only two ſtraight lines that are equal to one another, one upon each ſide of the ſhorteſt line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre: let the centre be E; of all the ſtraight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA is the greateſt, and FD, the other part of the diameter AD, is the leaſt: and of the others, FB is greater than FC, and FC than FG.

Join BE, CE, GE; and becauſe two ſides of a triangle are greater [a] than the third, BE, EF are greater than BF; but

[a] 20. 1.



but AE is equal to EB; therefore AE, EF, that is, AF, is greater than BF: again, becauſe BE is equal to CE, and FE common to the triangles BEF, CEF, the two ſides BE, EF are equal to the two CE, EF; but the angle BEF is greater than the angle CEF; there- b 24. 1. fore the baſe BF is greater [b] than the baſe FC: for the ſame reaſon, CF is greater than GF: again, becauſe GF, FE are greater [a] than EG, and EG is equal to ED; GF, FE are greater than ED: take away the common part FE, and the remainder GF is greater than the remainder FD: therefore FA is the greateſt, and FD the leaſt of all the ſtraight lines from F to the circumference; and BF is greater than CF, and CF than GF.

Alſo there can be drawn only two equal ſtraight lines from the point F to the circumference, one upon each ſide of the ſhorteſt line FD: at the point E in the ſtraight line EF, c 23. 1. make [c] the angle FEH equal to the angle GEF, and join FH: Then, becauſe GE is equal to EH, and EF common to the two triangles GEF, HEF; the two ſides GE, EF are equal to the two HE, EF; and the angle GEF is equal to the d 4. 1. angle HEF; therefore the baſe FG is equal [d] to the baſe FH: but beſides FH, no ſtraight line can be drawn from F to the circumference equal to FG: for, if there can, let it be FK; and becauſe FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which paſſes through the centre, is equal to one which is more remote, which is impoſſible. Therefore, if any point be taken, &c. Q. E. D.

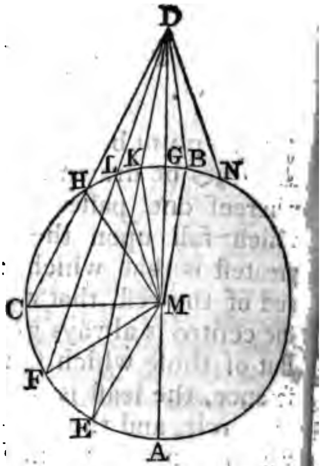
PROP.

PROP. VII. THEOR.

IF any point be taken without a circle, and ſtraight lines be drawn from it to the circumference, whereof one paſſes through the centre; of thoſe which fall upon the concave circumference, the greateſt is that which paſſes through the centre; and of the reſt, that which is nearer to that through the centre is always greater than the more remote: But of thoſe which fall upon the convex circumference, the leaſt is that between the point without the circle, and the diameter; and of the reſt, that which is nearer to the leaſt is always leſs than the more remote: And only two equal ſtraight lines can be drawn from the point unto the circumference, one upon each ſide of the leaſt.

Let ABC be a circle, and D any point without it, from which let the ſtraight lines DA, DE, DF, DC be drawn to the circumference, whereof DA paſſes through the centre. Of thoſe which fall upon the concave part of the circumference AEFC, the greateſt is AD which paſſes through the centre; and the nearer to it is always greater than the more remote, viz. DE than DF, and DF than DC: but of thoſe which fall upon the convex circumference HLKG, the leaſt is DG between the point D and the diameter AG; and the nearer to it is always leſs than the more remote, viz. DK than DL, and DL than DH.

Take [a] M the centre of the circle ABC, and join ME, MF, a 1. 3.
MC, MK, ML, MH: And becauſe AM is equal to ME, add
MD to each, therefore AD is equal to EM, MD; but EM,
MD are greater [b] than ED; therefore alſo AD is greater b 20. 1.
than ED. Again, becauſe ME is equal to MF, and MD
common

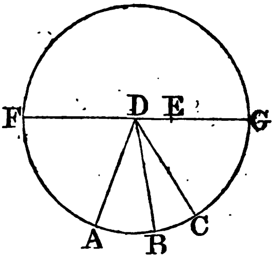


common to the triangles EMD, FMD; EM, MD are equal to FM, MD; but the angle EMD is greater than the angle FMD; therefore the baſe ED c 24. 1. is greater[c] than the baſe FD; In like manner it may be ſhewn that FD is greater than CD; Therefore DA is the greateſt; and DE greater than DF, and DF than DC. And becauſe MK, KD are greater[b] than MD, and MK is equal to MG, d 4. Ax. the remainder KD is greater[d] than the remainder GD, that is GD is leſs than KD: And becauſe MK, DK are drawn to the point K within the triangle MLD from M, D, the extremities of its ſide MD; MK, KD e 21. 1. are leſs[e] than ML, LD, whereof MK is equal to ML; therefore the remainder DK is leſs than the remainder DL: In like manner, it may be ſhewn that DL is leſs than DH: Therefore DG is the leaſt, and DK leſs than DL, and DL than DH. Alſo there can be drawn only two equal ſtraight lines from the point D to the circumference, one upon each ſide of the leaſt: at the point M, in the ſtraight line MD, make the angle DMB equal to the angle DMK, and join DB; and becauſe in the triangles KMD, BMD, the ſide KM is equal to the ſide BM, and MD common to both, and alſo the f 4. 1. angle KMD equal to the angle BMD, the baſe DK is equal[f] to the baſe DB. But, beſides DB, no ſtraight line can be drawn from D to the circumference, equal to DK: for, if there can, let it be DN; then, becauſe DN is equal to DK, and DK equal to DB, DB is equal to DN; that is, the nearer to DG, the leaſt, equal to the more remote, which has been ſhewn to be impoſſible. If, therefore, any point, &c. Q. E. D.

PROP. IX. PROB.

IF a point be taken within a circle, from which there fall more than two equal ſtraight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal ſtraight lines, viz. DA, DB, DC, the point D is the centre of the circle.



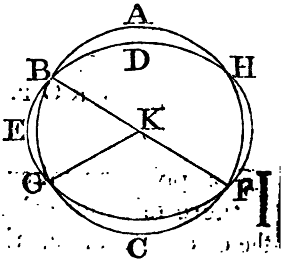
For, if not, let E be the centre, join DE and produce it to the circumference in F, G; then FG is a diameter of the circle ABC: And becauſe in FG, the diameter of the circle ABC, there is taken the point D which is not the centre, DG ſhall be the greateſt line from it to the circumference, and DC greater [a] than DB, and DB than DA; but they are likewiſe equal, which is impoſſible: Therefore E is not the centre of the circle ABC: In like manner, it may be demonſtrated, that no other point but D is the centre; D therefore is the centre. Wherefore, if a point be taken, &c. Q. E. D.

[a] 7. 3.

PROP. X. THEOR.

ONE circumference of a circle cannot cut another in more than two points.

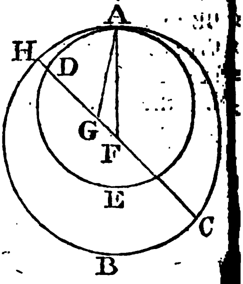
If



If it be poſſible, let the circumference FAB cut the circumference DEF in more than two points, viz. in B, G, F; take the centre K of the circle ABC, and join KB, KG, KF: and becauſe within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal ſtraight lines KB, a 9. 3. KG, KF, the point K is [a] the centre of the circle DEF; but K is alſo the centre of the circle ABC; therefore the ſame point is the centre of two circles that cut one another, which b 5. 3. is impoſſible [b]. Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D.

PROP. XI. THEOR.

IF two circles touch each other internally, the ſtraight line which joins their centres being produced, ſhall paſs through the point of contact.



Let the two circles ABC, ADE, touch each other internally in the point A, and let F be the centre of the circle ABC, and G the centre of the circle ADE; the ſtraight line which joins the centres F, G, being produced, paſſes through the point A.

For, if not, let it fall otherwiſe, if poſſible, as FGDH, and join AF, AG: and becauſe AG, GF are a 20. 1. greater [a] than FA, that is, than FH, for FA is equal to FH, both being from the ſame centre; take away the common part FG, and the remainder AG will be greater than the remainder GH. Bu AG is equal to GD, therefore GD is greater than GH; a it is alſo leſs, which is impoſſible. Therefore the ſtraight lin which joins the points F and G cannot fall otherwiſe than o the point A; that is, it muſt paſs through A. Therefore, two circles, &c. Q. E. D.

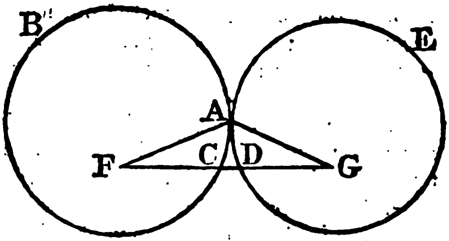
PROP.

PROP. XII. THEOR.

IF two circles touch each other externally, the ſtraight line which joins their centres ſhall paſs through the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE: The ſtraight line which joins the points F, G ſhall paſs through the point of contact A.

For, if not, let it paſs otherwiſe, if poſſible, as FCDG, and join FA, AG: and becauſe F is the centre of the circle ABC, AF is equal to FC: Alſo becauſe G is the centre of the circle ADE, AG is equal to GD: Therefore FA, AG are equal to FC, DG; wherefore the whole FG is greater than FA, AG; but it is alſo leſs[a], which is impoſſible: Therefore the ſtraight line which joins the points F, G ſhall not paſs otherwiſe than through the point of contact A; that is, it muſt paſs through it. Therefore, if two circles, &c. Q. E. D.



[a] 20. 1.

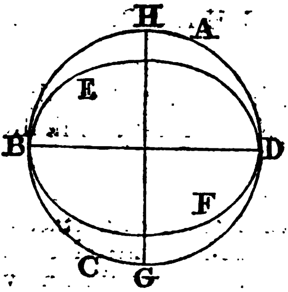
PROP. XIII. THEOR.

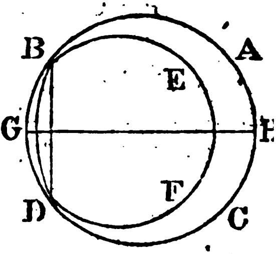
ONE circle cannot touch another in more points than one, whether it touches it on the inſide or outſide.

For, if it be poſſible, let the circle EBF touch the circle ABC in more points than one, and firſt on the inſide, in the points B, D; join BD, and draw[a] GH biſecting BD at right angles:

[a] 10. 11. 1.

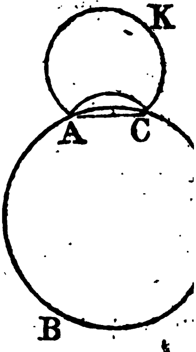
Book III. angles: Therefore, becauſe the points B, D are in the circ





ference of each of the circles, the ſtraight line BD falls wi
b 2. 3. each b of them; and their centres are c in the ſtraight line
c Cor 1. 3 which biſects BD at right angles: Therefore GH p
d 11. 3. through the point of contact d; but it does not paſs throug
becauſe the points B, D are without the ſtraight line
which is abſurd: Therefore one circle cannot touch ano
on the inſide in more points than one.

Nor can two circles touch one another on the outſid
more than one point: For, if it be poſſible, let the circle A
touch the circle ABC in the points A, C, and join A
Therefore, becauſe the two points A, C are in the circumference of the circle ACK, the ſtraight line AC
b 2. 3. which joins them ſhall fall within b the circle ACK: And the circle ACK is without the circle ABC; and therefore the ſtraight line AC is without this laſt circle; but, becauſe the points A, C are in the circumference of the circle ABC, the ſtraight line AC muſt be within b the ſame circle, which is abſurd: Therefore one circle cannot touch another on
the outſide in more than one point; and it has been ſhe
that they cannot touch on the inſide in more points than
Therefore, one circle, &c. Q. E. D.



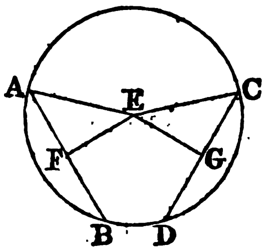
PROP. XIV. THEOR.

EQUAL ſtraight lines in a circle are equally diſtant from the centre; and thoſe which are equally diſtant from the centre, are equal to one another.

Let the ſtraight lines AB, CD, in the circle ABDC, be equal to one another; they are equally diſtant from the centre.

Take E the centre of the circle ABDC, and from it draw EF, EG perpendiculars to AB, CD: Then, becauſe the ſtraight line EF, paſſing through the centre, cuts the ſtraight line AB, which does not paſs through the centre, at right angles, it alſo biſects[a] it: Wherefore AF is equal to FB, and AB double of AF. For the ſame reaſon, CD is double of CG: And AB is equal to CD; therefore AF is equal to CG: And becauſe AE is equal to EC, the ſquare of AE is equal to the ſquare of EC: But the ſquares of AF, FE are equal[b] to the ſquare of AE, becauſe the angle AFE is a right angle; and, for the like reaſon, the ſquares of EG, GC are equal to the ſquare of EC: Therefore the ſquares of AF, FE are equal to the ſquares of CG, GE, of which the ſquare of AF is equal to the ſquare of CG, becauſe AF is equal to CG; therefore the remaining ſquare of FE is equal to the remaining ſquare of EG, and the ſtraight line EF is therefore equal to EG: But ſtraight lines in a circle are ſaid to be equally diſtant from the centre, when the perpendiculars drawn to them from the centre are equal[c]: Therefore AB, CD are equally diſtant from the centre.

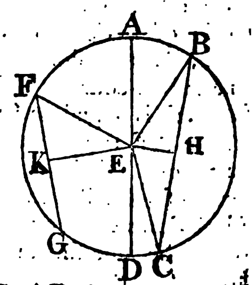
a 3. 3.
b 47. 1.
c 3. Def. 3.



Next, if the ſtraight lines AB, CD be equally diſtant from the centre, that is, if FE be equal to EG, AB is equal to CD: For, the ſame conſtruction being made, it may, as before, be demonſtrated, that AB is double of AF, and CD double o CG, and that the ſquares of EF, FA are equal to the ſquare of EG, GC; of which the ſquare of FE is equal to the ſquar of EG, becauſe FE is equal to EG; therefore the remainin ſquare of AF is equal to the remaining ſquare of CG; and th ſtraight line AF is therefore equal to CG: And AB is doubl of AF, and CD double of CG; wherefore AB is equal to CD Therefore equal ſtraight lines, &c. Q. E. D.

PROP. XV. THEOR.

THE diameter is the greateſt ſtraight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the leſs.



Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG; AD is greater than any ſtraight line BC which is not a diameter, and BC greater than FG.

From the centre draw EH, EK perpendiculars to BC, FG, and join EB, EC, EF; and becauſe AE is equal to EB, and ED to EC, AD is equal to E EC: But EB, EC, are greater a than BC; wherefore, al AD is greater than BC.

a 10. 1.

And, becauſe BC is nearer to the centre than FG, EH less b than EK: But, as was demonſtrated in the preced BC is double of BH, and FG double of FK, and the ſquares EH, HB are equal to the ſquares of EK, KF, of which th ſquare of EH is leſs than the ſquare of EK, becauſe EH is le than EK; therefore the ſquare of BH is greater than the ſqu of FK, and the ſtraight line BH greater than FK; and there fore BC is greater than FG.

b 4. Def. 3.

Next, Let BC be greater than FG; BC is nearer to th centre than FG; that is, the ſame conſtruction being ma

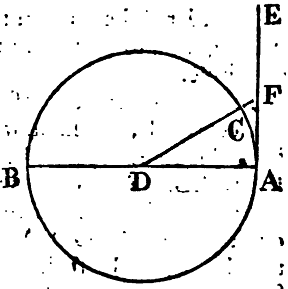
EH is leſs than EK: Becauſe BC is greater than EG, BH likewiſe is greater than KF; and the ſquares of BH, HE are equal to the ſquares of FK, KE, of which the ſquare of BH is greater than the ſquare of FK, becauſe BH is greater than FK; therefore the ſquare of EH is leſs than the ſquare of EK, and the ſtraight line EH leſs than EK. Wherefore the diameter, &c. Q. E. D.

PROP. XVI. THEOR.

THE ſtraight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no ſtraight line can be drawn between that ſtraight line and the circumference from the extremity, ſo as not to cut the circle.

Let ABC be a circle, the centre of which is D, and the diameter AB; and let AE be drawn from A perpendicular to AB, AE ſhall fall without the circle.

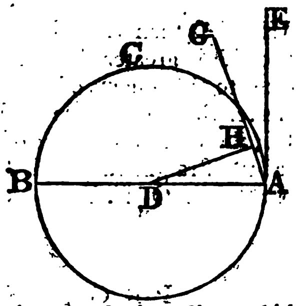
In AE take any point F, join DF, and let DF meet the circle in C. Becauſe DAF is a right angle, it is greater than the angle AFD[a]; but the greateſt angle of any triangle is ſubtended by the greater ſide[b], therefore DF is greater than DA; and DA is equal to DC, therefore DF is greater than DC, and the point F is therefore without the circle. Now F is any point whatever in the line AE, therefore AE falls without the circle.



a. 32. 1.

b 19. 1.

Again, between the ſtraight line AE and the circumference no ſtraight line can be drawn from the point A, which does not cut the circle. Let AG be drawn, making the angle DAG leſs than a right angle; from D draw DH at right angles to AG; and becauſe the angle DHA is a right angle,



angle, each of the other angles of the triangle DAH is leſs than a right angle [a]; the angle DAH is therefore leſs than the angle DHA, and therefore alſo the ſide DH is leſs than the ſide DA [b]. The point H, therefore, is within the circle, and therefore the ſtraight line AG cuts the circle.

Cor. From this it is manifeſt that the ſtraight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, becauſe, if it did meet the circle in two, it would fall within it [c]. 'Alſo it is evident that there can be but one ſtraight line which touches the circle in the ſame point.'

c 2. 3.

PROP. XVII. PROB.

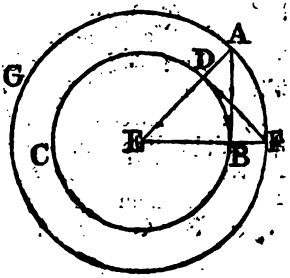
TO draw a ſtraight line from a given point, either without or in the circumference, which ſhall touch a given circle.

Firſt, let A be a given point without the given circle BCD; it is required to draw a ſtraight line from A which ſhall touch the circle.

a 1. 3.

Find [a] the centre E of the circle, and join AE; and from the centre E, at the diſtance EA, deſcribe the circle AFG; from the point D draw [b] DF at right angles to EA, and join EBF, AB. AB touches the circle BCD.

b 11. 1.



Becauſe E is the centre of the circles BCD, AFG, EA is equal to EF: and ED to EB; therefore the two ſides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the baſe DF is equal to the baſe AB, and the triangle FED to the triangle AEB, and the other angles to the other angles c: Therefore the angle EBA is equal to the angle EDF; but EDF is a right angle, wherefore EBA is a right angle; and EB is drawn from the centre: but a ſtraight line drawn from the extremity of a diameter, at right angles to it, touches the circle d: Therefore AB touches the circle; and it is drawn from the given point A. Which was to be done.

c. 4. 1.

d Cor. 16. 3.

But, if the given point be in the circumference of the circle, as the point D, draw DE to the centre E and DF at right angles to DE; DF touches the circle d.

PROP. XVIII. THEOR.

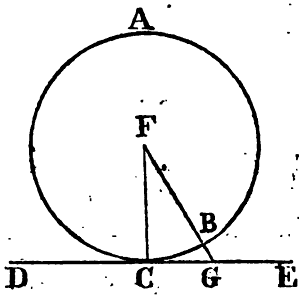
IF a ſtraight line touches a circle, the ſtraight line drawn from the centre to the point of contact, ſhall be perpendicular to the line touching the circle.

Let the ſtraight line DE touch the circle ABC in the point C; take the centre F, and draw the ſtraight line FC: FC is perpendicular to DE.

For, if it be not, from the point F draw FBG perpendicular to DE; and becauſe FGC is a right angle, GCF is b an acute angle; and to the greater angle the greateſt c ſide is oppoſite: Therefore FC is greater than FG; but FC is equal to FB; therefore FB is greater than FG, the leſs than the greater, which is impoſſible; wherefore FG is not perpendicular to DE: In the ſame manner it may be ſhewn, that no other is perpendicular to it beſides FC, that is, FC is perpendicular to DE. Therefore, if a ſtraight line, &c. Q. E. D.

b 17. 1.

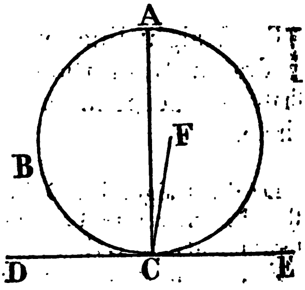
c 19. 1.



PROP. XIX. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn at right angles to the touching line, the centre of the circle ſhall be in that line.

Let the ſtraight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the centre of the circle is in CA.



For, if not, let F be the centre, if poſſible, and join CF: Becauſe DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular[a] to DE; therefore FCE is a right angle: But ACE is alſo a right angle; therefore the angle FCE is equal to the angle ACE, the leſs to the greater, which is impoſſible: Wherefore F is not the centre of the circle ABC: In the ſame manner, it may be ſhewn, that no other point which is not in CA, is the centre; that is, the centre is in CA. Therefore, if a ſtraight line, &c. Q. E. D.

a 18. 3.

PROP. XX. THEOR.

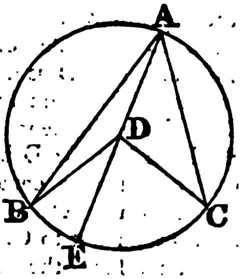
THE angle at the centre of a circle is double of the angle at the circumference, upon the ſame baſe, that is, upon the ſame part of the circumference.

Let ABC be a circle, and BDC an angle at the centre, and BAC an angle at the circumference, which have the ſame circumference BC for their baſe; the angle BDC is double of the angle BAC.

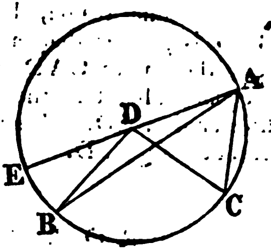
Firſt, let D, the centre of the circle, be within the angle BAC, and join AD, and produce it to E: Becauſe DA is equal to DB, the angle DAB is equal[a] to the angle DBA; therefore the angles DAB, DBA are double of the angle DAB; but the angle BDE is equal[b] to the angles DAB, DBA; therefore alſo the angle BDE is double of the angle DAB: For the ſame reaſon, the angle EDC is double of the angle DAC: Therefore the whole angle BDC is double of the whole angle BAC.

a 5. 1.

b 32. 1.



Again, let D, the centre of the circle be without the angle BAC, and join AD and produce it to E. It may be demonſtrated, as in the firſt caſe, that the angle EDC is double of the angle DAC, and that EDB a part of the firſt is double of DAB a part of the other; therefore the remaining angle BDC is double of the remaining angle BAC. Therefore the angle at the centre, &c. Q. E. D.

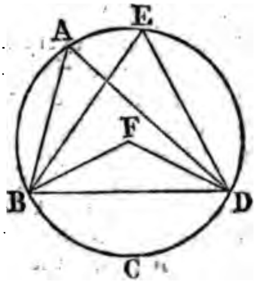


PROP. XXI. THEOR.

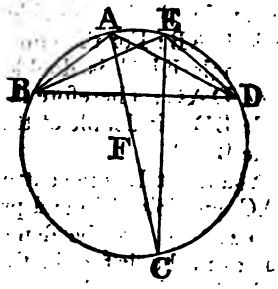
THE angles in the ſame ſegment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED angles in the ſame ſegment BAED: The angles BAD, BED are equal to one another.

Take F the centre of the circle ABCD: And, firſt, let the ſegment BAED be greater than a ſemicircle, and join BF, FD: And becauſe the angle BFD is at the centre, and the angle



BAD at the circumference, and that they have the ſame part of the circumference, viz. BCD, for their baſe; therefore the a 20. 3. angle BFD is double [a] of the angle BAD: for the ſame reaſon, the angle BFD is double of the angle BED: Therefore the angle BAD is equal to the angle BED.

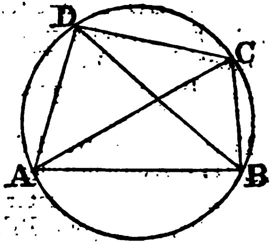


But, if the ſegment BAED be not greater than a ſemicircle, let BAD, BED be angles in it; theſe alſo are equal to one another: Draw AF to the centre, and produce it to C, and join CE: Therefore the ſegment BADC is greater than a ſemicircle; and the angles in it BAC, BEC are equal, by the firſt caſe: For the ſame reaſon, becauſe CBED is greater than a ſemicircle, the angles CAD, CED are equal: Therefore the whole angle BAD is equal to the whole angle BED. Wherefore the angles in the ſame ſegment, &c. Q. E. D.

PROP. XXII. THEOR.

THE oppoſite angles of any quadrilateral figure deſcribed in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its oppoſite angles are together equal to two right angles.

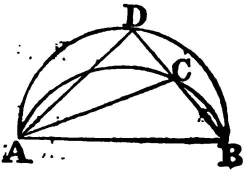


Join AC, BD; and becauſe the three angles of every triangle a 32. 1. are equal [a] to two right angles, the three angles of the triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles: But the angle CAB b 21. 3. is equal [b] to the angle CDB, becauſe they are in the ſame ſegment BADC, and the angle ACB is equal to the angle

ADB, becauſe they are in the ſame ſegment ADCB: Therefore the whole angle ADC is equal to the angles CAB, ACB: To each of theſe equals add the angle ABC; therefore the angles ABC, CAB, BCA, are equal to the angles ABC, ADC: But ABC, CAB, BCA are equal to two right angles; therefore alſo the angles ABC, ADC are equal to two right angles: In the ſame manner, the angles BAD, DCB may be ſhewn to be equal to two right angles. Therefore, the oppoſite angles, &c. Q. E. D.

PROP. XXIII. THEOR.

UPON the ſame ſtraight line, and upon the ſame ſide of it, there cannot be two ſimilar ſegments of circles, not coinciding with one another.



If it be poſſible, let the two ſimilar ſegments of circles, viz. ACB, ADB, be upon the ſame ſide of the ſame ſtraight line AB, not coinciding with one another: then, becauſe the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point [a]: one of the ſegments muſt therefore fall within the other: let ACB fall within ADB, and draw the ſtraight line BCD, and join CA, DA: and becauſe the ſegment ACB is ſimilar to the ſegment ADB, and that ſimilar ſegments of circles contain [b] equal angles; the angle ACB is equal to the angle ADB, the exterior to the interior, which is impoſſible [c]. Therefore, there cannot be two ſimilar ſegments of a circle upon the ſame ſide of the ſame line, which do not coincide. Q. E. D.

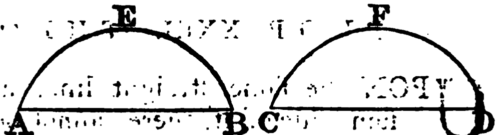
a 10. 3.
b 9. def. 3.
c 16. 1.

Book III.

PROP. XXIV. THEOR.

SIMILAR ſegments of circles upon equal ſtraight lines, are equal to one another.

Let AEB, CFD be ſimilar ſegments of circles upon the equal ſtraight lines AB, CD; the ſegment AEB is equal to the ſegment CFD.



For, if the ſegment AEB be applied to the ſegment CFD, ſo as the point A be on C, and the ſtraight line AB upon CD, the point B ſhall coincide with the point D, becauſe AB is equal to CD: Therefore the ſtraight line AB coinciding with CD, the ſegment AEB muſt [a] coincide with the ſegment CFD, and therefore is equal to it. Wherefore, ſimilar ſegments, &c. Q. E. D.

a 23. 3.

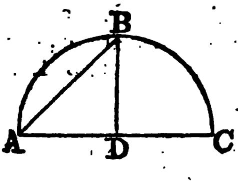
PROP. XXV. PROB.

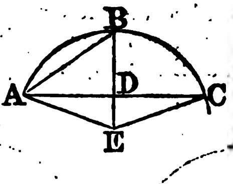
A Segment of a circle being given, to deſcribe the circle of which it is the ſegment.

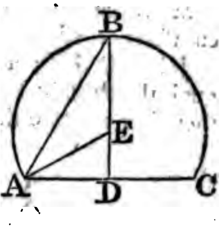
Let ABC be the given ſegment of a circle; it is required to deſcribe the circle of which it is the ſegment.

a 10. 1. b 11. 1. c 6. 1. d 9. 3.

Biſect [a] AC in D, and from the point D draw [b] DB at right angles to AC, and join AB: Firſt, let the angles ABD, BAD be equal to one another; then the ſtraight line BD is equal [c] to DA, and therefore to DC; and becauſe the three ſtraight lines DA, DB, DC, are all equal; D is the centre of the circle [d]: from the centre D, at the diſtance of any of the three DA, DB, DC, deſcribe a circle; this ſhall paſs through the other points; and the circle of which ABC is a ſegment is deſcribed: and becauſe the centre D is in AC, the ſegment ABC is a ſemicircle; but if the angles ABD, BAD are not equal to one another, at the point A, in the ſtraight line AB make [e] the angle BAE equal to the angle ABD, and produce BD, if neceſſary, to E, and join EC: e 23. 1.







and becauſe the angle ABE is equal to the angle BAE, the ſtraight line BE is equal [e] to EA: and becauſe AD is equal to DC, and DE common to the triangles ADE, CDE, the two ſides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefore the baſe AE is equal [f] to the baſe EC: but AE was ſhewn to be equal to f 4. 1. EB, wherefore alſo BE is equal to EC: and the three ſtraight lines AE, EB, EC are therefore equal to one another; wherefore [d] E is the centre of the circle. From the centre d 9. 3. E, at the diſtance of any of the three AE, EB, EC, deſcribe a circle, this ſhall paſs through the other points; and the circle of which ABC is a ſegment is deſcribed: and it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the ſegment ABC, which therefore is leſs than a ſemicircle: but if the angle ABD be leſs than BAD, the centre E falls within the ſegment ABC, which is therefore greater than a ſemicircle: Wherefore a ſegment of a circle being given, the circle is deſcribed of which it is a ſegment. Which was to be done.

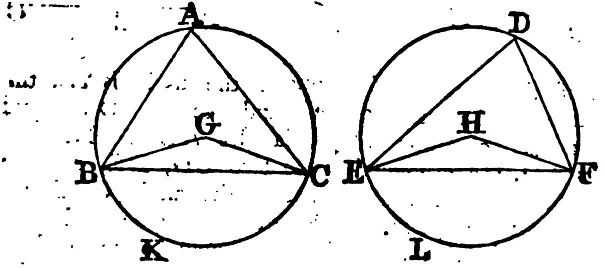
PROP. XXVI. THEOR.

IN equal circles, equal angles ſtand upon equal circumferences, whether they be at the centres or circumferences.

Let

Book III. Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences: the circumference BKC is equal to the circumference ELF.

Join BC, EF; and becauſe the circles ABC, DEF are equal, the ſtraight lines drawn from their centres are equal: therefore the two ſides BG, GC, are equal to the two EH,



HF; and the angle at G is equal to the angle at H; there-
a 4. 1. fore the baſe BC is equal [a] to the baſe EF: and becauſe the
angle at A is equal to the angle at D, the ſegment BAC is
b 9. def. 3. ſimilar [b] to the ſegment EDF; and they are upon equal
ſtraight lines BC, EF; but ſimilar ſegments of circles upon
c 24. 3. equal ſtraight lines are equal [c] to one another, therefore the
ſegment BAC is equal to the ſegment EDF: but the whole
circle ABC is equal to the whole DEF; therefore the remaining ſegment BKC is equal to the remaining ſegment ELF, and the circumference BKC to the circumference ELF. Wherefore, in equal circles, &c. Q. E. D.

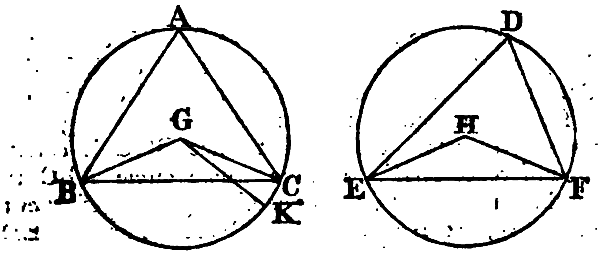
PROP. XXVII. THEOR.

IN equal circles, the angles which ſtand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF ſtand upon the equal circumferences BC, EF: the angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be equal to the angle EHF, it is manifeſt [a] that the angle BAC is alſo equal to EDF. But, if not, one of them is the greater: let BGC be the greater, and at the point G, in the ſtraight line BG, make [b] the angle BGK equal to the angle EHF; but equal angles ſtand upon equal circumferences [c], when they are at the centre; therefore the circumference BK is equal to the circumference EF: but EF is equal to BC; therefore alſo BK is equal to BC, the leſs to the greater, which is impoſſible: therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it;

a 20. 3.
b 23. 1.
c 26. 3.



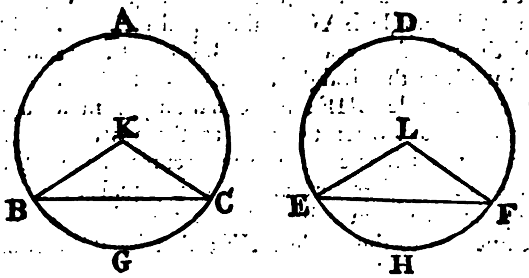
and the angle at A is half of the angle BGC, and the angle at D half of the angle EHF: therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVIII. THEOR.

IN equal circles, equal ſtraight lines cut off equal circumferences, the greater equal to the greater, and the leſs to the leſs.

Let ABC, DEF be equal circles, and BC, EF equal ſtraight lines in them, which cut off the two greater circumferences BAC, EDF, and the two leſs BGC, EHF: the greater BAC is equal to the greater EDF, and the leſs BGC to the leſs EHF.

Take [a] K, L, the centres of the circles, and join BK, KC,
a 1. 3. EL, LF: and becauſe the circles are equal, the ſtraight lines



from their centres are equal; therefore BK, KC are equal to
EL, LF; and the baſe BC is equal to the baſe EF; there-
b 8. 1. fore the angle BKC is equal [b] to the angle ELF: but equal
c 26. 3. angles ſtand upon equal [c] circumferences, when they are at
the centres; therefore the circumference BGC is equal to
the circumference EHF. But the whole circle ABC is e-
qual to the whole EDF; the remaining part therefore of the
circumference, viz. BAC, is equal to the remaining part EDF.
Therefore, in equal circles, &c. Q. E. D.

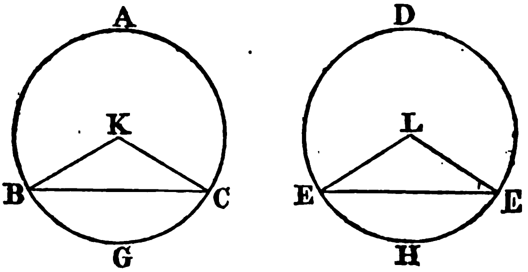
PROP. XXIX. THEOR.

IN equal circles equal circumferences are ſubtended by equal ſtraight lines.

Let ABC, DEF be equal circles, and let the circumferences BGC, EHF alſo be equal; and join BC, EF: the ſtraight line BC is equal to the ſtraight line GF.

a 1. 3. Take [a] K, L the centres of the circles, and join BK, KC,
EL, LF: and becauſe the circumference BGC is equal to the
b 27. 3. circumference EHF, the angle BKC is equal [b] to the angle
ELF: and becauſe the circles ABC, DEF are equal, the
ſtraight lines from their centres are equal: therefore BK, KC

are equal to EL, LF, and they contain equal angles: therefore the baſe BC is equal [c] to the baſe EF. Therefore, in equal circles, &c. Q. E. D. c 4. 1.



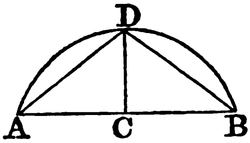
PROP. XXX. PROB.

To biſect a given circumference, that is, to divide it into two equal parts.

Let ADB be the given circumference; it is required to biſect it.

Join AB, and biſect [a] it in C; from the point C draw CD at right angles to AB, and join AD, DB: the circumference ADB is biſected in the point D. a 10. 1.

Becauſe AC is equal to CB, and CD common to the triangles ACD, BCD, the two ſides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, becauſe each of them is a right angle; therefore the baſe AD is equal [b] to the baſe BD. b 4. 1. But equal ſtraight lines cut off equal [c] circumferences, the greater equal to the greater, and the leſs to the leſs, and AD, DB are each of them leſs than a ſemicircle; becauſe DC paſſes through the centre [d]: Wherefore the circumference AD is equal to the circumference DB: therefore the given circumference is biſected in D. Which was to be done. c 28. 3. d Cor. 1. 3.

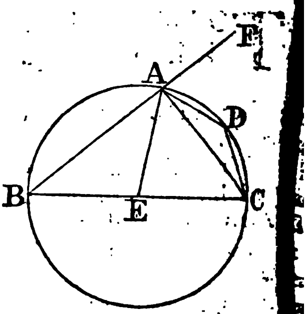


PROP. XXXI. THEOR.

IN a circle, the angle in a ſemicircle is a right angle; but the angle in a ſegment greater than a ſemicircle is leſs than a right angle; and the angle in a ſegment leſs than a ſemicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA dividing the circle into the ſegments ABC, ADC, and join BA, AD, DC; the angle in the ſemicircle BAC is a right angle; and the angle in the ſegment ABC, which is greater than a ſemicircle, is leſs than a right angle; and the angle in the ſegment ADC, which is leſs than a ſemicircle, is greater than a right angle.

Join AE, and produce BA to F; and becauſe BE is equal
a 5. 1. to EA, the angle EAB is equal [a] to EBA; alſo, becauſe AE is equal to EC the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB. But FAC, the exterior angle of the triangle ABC, is alſo equal [b] to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is
c 7. def. 1. therefore a right [c] angle: wherefore the angle BAC in a ſemicircle is a right angle.



And becauſe the two angles ABC, BAC of the triangle
d 17. 1. ABC are together leſs [d] than two right angles, and that BAC is a right angle, ABC muſt be leſs than a right angle; and therefore the angle in a ſegment ABC greater than a ſemicircle, is leſs than a right angle.

And becauſe ABCD is a quadrilateral figure in a circle,
e 22. 3. any two of its oppoſite angles are equal [e] to two right angles; therefore the angles ABC, ADC are equal to two right angles; and ABC is leſs than a right angle; wherefore the other ADC is greater than a right angle.

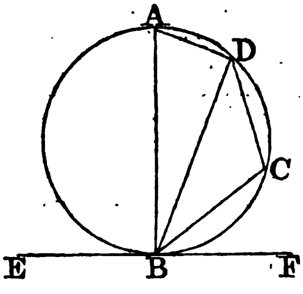
Cor. From this it is manifeſt, that if one angle of a triangle be equal to the other two, it is a right angle, becauſe the angle adjacent to it is equal to the ſame two; and when the adjacent angles are equal, they are right angles.

PROP. XXXII. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn cutting the circle, the angles made by this line with the line touching the circle, ſhall be equal to the angles which are in the alternate ſegments of the circle.

Let the ſtraight line EF touch the circle ABCD in B, and from the point B let the ſtraight line BD be drawn cutting the circle: The angles which BD makes with the touching line EF ſhall be equal to the angles in the alternate ſegments of the circle: that is, the angle FBD is equal to the angle which is in the ſegment DAB, and the angle DBE to the angle in the ſegment BCD.

From the point B draw [a] BA at right angles to EF, and a 11. 1.
take any point C in the circumference BD, and join AD, DC,
CB; and becauſe the ſtraight line EF touches the circle ABCD
in the point B, and BA is
drawn at right angles to the
touching line from the point of
contact B, the centre of the
circle is [b] in BA; therefore b 19. 3.
the angle ADB in a ſemicircle
is a right [c] angle, and conſe- c 31. 3.
quently the other two angles
BAD, ABD are equal [d] to a
right angle: but ABF is like- d 32. 1.
wiſe a right angle; therefore
the angle ABF is equal to the angles BAD, ABD: take



from theſe equals the common angle ABD; therefore the remaining angle DBF is equal to the angle BAD, which is in the alternate ſegment of the circle: and becauſe ABCD is a quadrilateral figure in a circle, the oppoſite angles BAD, BCD are equal [e] to two right angles; therefore the angles DBF, DBE, being likewiſe equal [f] to two right angles, are equal to the angles BAD, BCD; and DBF has been proved equal to BAD: therefore the remaining angle DBE is equal to the angle BCD in the alternate ſegment of the circle.

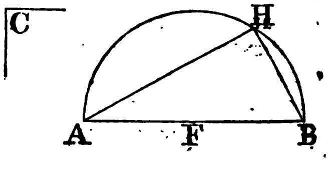
e 22. 3.
f 13. 1.

Wherefore, if a ſtraight line, &c. Q. E. D.

PROP. XXXIII. PROB.

UPON a given ſtraight line to deſcribe a ſegment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and the angle at C the given rectilineal angle; it is required to deſcribe upon the given ſtraight line AB a ſegment of a circle, containing an angle equal to the angle C.

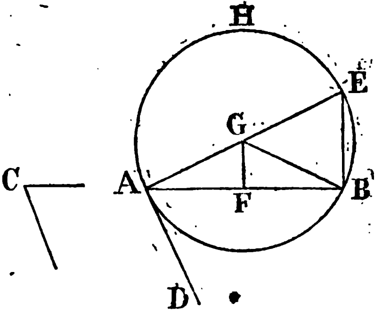


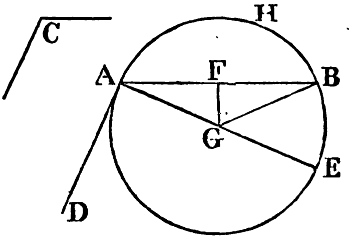
First, let the angle at C be a right angle, and biſect [a] AB in F, and from the centre F, at the diſtance FB, deſcribe the ſemicircle AHB; therefore the angle AHB in a ſemicircle is [b] equal to the right angle at C.

a 10. 1.
b 31. 3.

But, if the angle C be not a right angle, at the point A, in the ſtraight line AB, make [c] the angle BAD equal to the angle

c 23. 1.





angle C, and from the point A draw [d] AE at right angles to AD; biſect [a] AB in F, and from F draw [d] FG at right angles to AB, and join GB: And becauſe AF is equal to FB, and FG common to the triangles AFG, BFG the two ſides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the baſe AG is equal [e] to the baſe GB; and the circle deſcribed from the centre G, at the diſtance GA, ſhall paſs through the point B; let this be the circle AHB: And becauſe from the point A the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD [f] touches the circle; and becauſe AB drawn from the point of contact A cuts the circle, the angle DAB is equal to the angle in the alternate ſegment AHB [g]: but the angle DAB is equal to the angle C, therefore alſo the angle C is equal to the angle in the ſegment AHB: Wherefore, upon the given ſtraight line AB the ſegment AHB of a circle is deſcribed which contains an angle equal to the given angle at C. Which was to be done.

d 11. 1.
e 4. 1.
f Cor. 16. 3.
g 32. 3.

PROP. XXXIV. PROB.

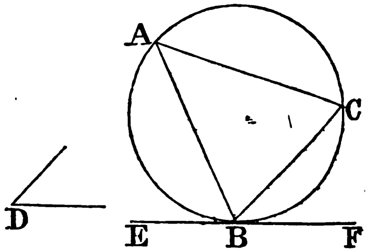
TO cut off a ſegment from a given circle which ſhall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a ſegment from the circle ABC that ſhall contain an angle equal to the angle D.

a 17. 3. Draw [a] the ſtraight line EF touching the circle ABC in the point B, and at the point B, in the ſtraight

b 23. 1. line BF make [b] the angle FBC equal to the angle D; therefore, becauſe the ſtraight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is

c 32. 3. equal [c] to the angle in the alternate ſegment BAC of the circle: but the angle FBC is equal to the angle D; therefore the angle in the ſegment BAC is equal to the angle D: wherefore the ſegment BAC is cut off from the given circle ABC containing an angle equal to the given angle D. Which was to be done.

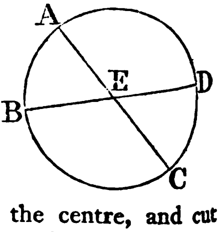


PROP. XXXV. THEOR.

IF two ſtraight lines within a circle cut one another, the rectangle contained by the ſegments of one of them is equal to the rectangle contained by the ſegments of the other.

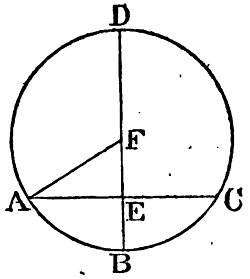
Let the two ſtraight lines AC, BD, within the circle ABCD, cut one another in the point E: the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED.

If AC, BD paſs each of them through the centre, ſo that E is the centre; it is evident, that AE, EC, BE, ED, being all equal, the rectangle AE, EC is likewiſe equal to the rectangle BE, ED.

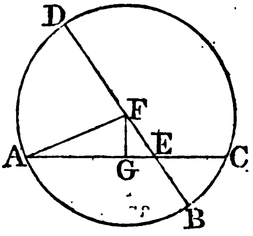


But let one of them BD paſs through the centre, and cut the other AC, which does not paſs through the centre, at

right angles, in the point E: then, if BD be biſected in F, F
is the centre of the circle ABCD; join AF: and becauſe
BD, which paſſes through the centre, cuts the ſtraight line
AC, which does not paſs through the
centre, at right angles in E, AE,
EC are equal[a] to one another: and a 3. 3.
becauſe the ſtraight line BD is cut
into two equal parts in the point F,
and into two unequal in the point E,
the rectangle BE, ED, together with
the ſquare of EF, is equal[b] to the b 5. 2.
ſquare of FB; that is, to the ſquare
of FA; but the ſquares of AE, EF
are equal[c] to the ſquare of FA; c 47. 1.
therefore the rectangle BE, ED, together with the ſquare of
EF, is equal to the ſquares of AE, EF: take away the common
ſquare of EF, and the remaining rectangle BE, ED is
equal to the remaining ſquare of AE; that is, to the rectangle
AE, EC.

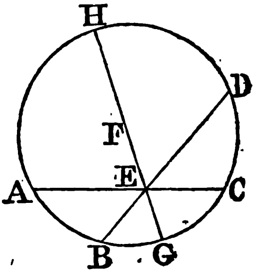


Next, Let BD, which paſſes through the centre, cut the other
AC, which does not paſs through the centre, in E, but
not at right angles: then, as before, if BD be biſected in F,
F is the centre of the circle. Join AF, and from F draw
[d] FG perpendicular to AC; therefore AG is equal[a] to GC; d 12. 1.
wherefore, the rectangle AE, EC, together with the ſquare
of EG, is equal[b] to the ſquare of AG: to each of theſe equals
add the ſquare of GF; therefore the rectangle AE,
EC, together with the ſquares of EG, GF, is equal to
the ſquares of AG, GF: But the
ſquares of EG, GF are equal[c] to
the ſquare of EF; and the ſquares
of AG, GF are equal to the
ſquare of AF: therefore the rectangle
AE, EC, together with the
ſquare of EF, is equal to the
ſquare of AF; that is, to the
ſquare of FB: but the ſquare of
FB is equal[b] to the rectangle BE,
ED, together with the ſquare of EF; therefore the rectangle
AE, EC, together with the ſquare of EF, is equal to the
rectangle BE, ED, together with the ſquare of EF: take away



Book III. way the common ſquare of EF, and the remaining rectangle AE, EC, is therefore equal to the remaining rectangle BE, ED.

Laſtly, Let neither of the ſtraight lines AC, BD paſs through the centre: take the centre F, and through E, the interſection of the ſtraight lines AC, DB, draw the diameter GEFH: and becauſe the rectangle AE, EC is equal, as has been ſhown, to the rectangle GE, EH; and, for the ſame reaſon, the rectangle BE, ED is equal to the ſame rectangle GE, EH; therefore the rectangle AE, EC is equal to the rectangle BE, ED. Wherefore, if two ſtraight lines, &c. Q. E. D.



PROP. XXXVI. THEOR.

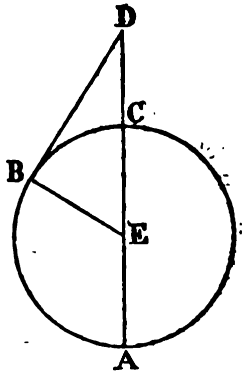
IF from any point without a circle two ſtraight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, ſhall be equal to the ſquare of the line which touches it.

Let D be any point without the circle ABC, and DCA, DB two ſtraight lines drawn from it, of which DCA cuts the circle, and DB touches the ſame: the rectangle AD, DC is equal to the ſquare of DB.

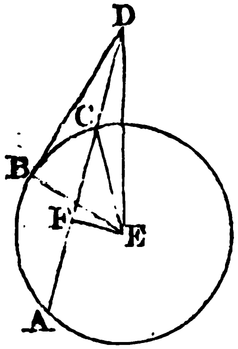
Either DCA paſſes through the centre, or it does not; firſt, Let it paſs through the centre E, and join EB; therefore the angle EBD is a right [a] angle: and becauſe the ſtraight

a 18. 3

t line AC is biſected in E, and
:ed to the point D, the rect-
AD, DC, together with the
of EC, is equal [b] to the ſquare b 6. 2.
); and CE is equal to EB;
re the rectangle AD, DC,
er with the ſquare of EB, is
:o the ſquare of ED: but the
of ED is equal [c] to the c 47. 1.
; of EB, BD, becauſe EBD is
: angle: therefore the rectangle
DC, together with the ſquare
, is equal to the ſquares of EB,
take away the common ſquare
; therefore the remaining rectangle AD, DC is equal
ſquare of the tangent DB.

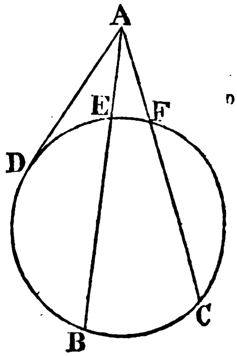


if DCA does not paſs through the centre of the circle
take [d] the centre E, and draw EF perpendicular [e] to d 1. 3.
nd join EB, EC, ED: and becauſe the ſtraight line e 12. 1.
hich paſſes through the centre, cuts the ſtraight line
which does not paſs through the
, at right angles, it ſhall like-
iſect [f] it; therefore AF is e- f 3. 3.
FC: and becauſe the ſtraight
C is biſected in F, and produ-
D, the rectangle AD, DC,
er with the ſquare of FC, is
to the ſquare of FD: to each
ſe equals add the ſquare of
therefore the rectangle AD,
together with the ſquares of
E, is equal to the ſquares of
E: but the ſquare of ED is
: to the ſquares of DF, FE be-
EFD is a right angle; and the ſquare of EC is equal
ſquares of CF, FE; therefore the rectangle AD, DC,
er with the ſquare of EC, is equal to the ſquare of
and CE is equal to EB; therefore the rectangle AD,
ogether with the ſquare of EB, is equal to the ſquare of
but the ſquares of EB, BD are equal to the ſquare [c] of
ecauſe EBD is a right angle; therefore the rectangle
DC, together with the ſquare of EB, is equal to the



Book III. ſquares of EB, BD: take away the common ſquare of EB; therefore the remaining rectangle AD, DC is equal to the ſquare of DB. Wherefore, if from any point, &c. Q. E. D.

Cor. If from any point without a circle, there be drawn two ſtraight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AE to the rectangle CA, AF: for each of them is equal to the ſquare of the ſtraight line AD which touches the circle.



PROP. XXXVII. THEOR.

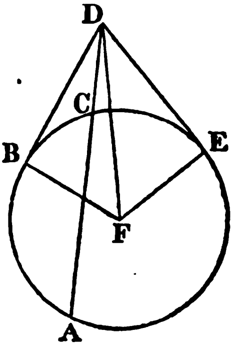
IF from a point without a circle there be drawn two ſtraight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the ſquare of the line which meets it, the line which meets ſhall touch the circle.

Let any point D be taken without the circle ABC, and from it let two ſtraight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD, DC, be equal to the ſquare of DB; DB touches the circle.

a 17. 3. Draw [a] the ſtraight line DE touching the circle ABC, find the centre F, and join FE, FB, FD; then FED is a
b 18. 3. right [b] angle: and becauſe DE touches the circle ABC, and
c 36. 3. DCA cuts it, the rectangle AD, DC is equal [c] to the ſquare of DE; but the rectangle AD, DC is, by hypotheſis, equal to the ſquare of DB: therefore the ſquare of DE is equal to the ſquare of DB; and the ſtraight line DE equal to the

Book III.

t line DB: and FE is equal to FB, wherefore DE, e equal to DB, BF; and the D is common to the two les DEF, DBF; therefore gle DEF is equal [d] to the DBF; and DEF is a right therefore alſo DBF is a right but FB, if produced, is a ter, and the ſtraight line is drawn at right angles to a ter, from the extremity of it, s [e] the circle: therefore DB s the circle ABC. Where- f from a point, &c. Q. E. D.



[d] 8. 1.

[e] 16. 3.

ELE-

ELEMENTS

OF

GEOMETRY.

BOOK IV.

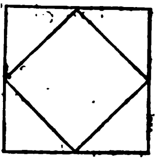
DEFINITIONS.

I.

A Rectilineal figure is ſaid to be inſcribed in another rectilineal figure, when all the angles of the inſcribed figure are upon the ſides of the figure in which it is inſcribed, each upon each.

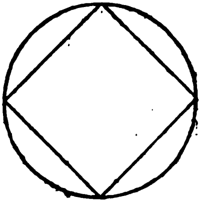
II.

In like manner, a figure is ſaid to be deſcribed about another figure, when all the ſides of the circumſcribed figure paſs through the angular points of the figure about which it is deſcribed, each through each.



III.

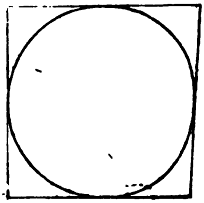
A rectilineal figure is ſaid to be inſcribed in a circle, when all the angles of the inſcribed figure are upon the circumference of the circle.



IV.

IV.

A rectilineal figure is ſaid to be deſcribed about a circle, when each ſide of the circumſcribed figure touches the circumference of the circle.

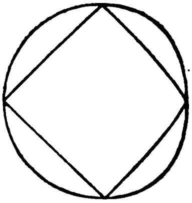


V.

In like manner, a circle is ſaid to be inſcribed in a rectilineal figure, when the circumference of the circle touches each ſide of the figure.

VI.

A circle is ſaid to be deſcribed about a rectilineal figure, when the circumference of the circle paſſes through all the angular points of the figure about which it is deſcribed.



VII.

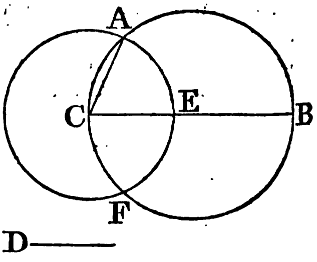
A ſtraight line is ſaid to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

IN a given circle to place a ſtraight line, equal to a given ſtraight line not greater than the diameter of the circle.

Let ABC be the given circle, and D the given ſtraight line, not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a ſtraight line BC is placed equal to D: But, if it is not, BC is greater than D; make CE equal[a] to D, and from the centre C, at the diſtance CE, deſcribe the circle AEF, and join CA: Therefore, becauſe C is the centre of the



a 3. 1.

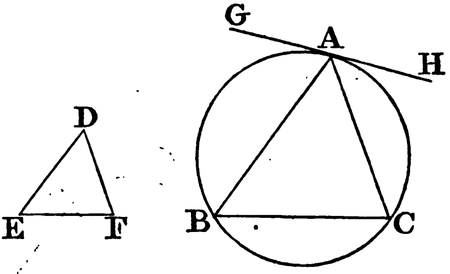
circle AEF, CA is equal to CE; but D is equal to CE; therefore D is equal to CA: Wherefore, in the circle ABC, a ſtraight line is placed, equal to the given ſtraight line D, which is not greater than the diameter of the circle. Which was to be done.

PROP. II. PROB.

IN a given circle to inſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inſcribe in the circle ABC a triangle equiangular to the triangle DEF.

Draw [a] the ſtraight line GAH touching the circle in the a 17. 3.
point A, and at the point A, in the ſtraight line AH, make [b] b 23. 1.
the angle HAC equal to the angle DEF; and at the point A, in the ſtraight line AG, make the angle GAB equal to the angle DFE, and join BC: Therefore, becauſe HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is equal [c] to the angle c 32. 3.
ABC in the alternate ſegment of the circle: But HAC is equal to the angle DEF; therefore alſo the angle ABC is equal to DEF: for the ſame reaſon, the angle ACB is equal to the angle DFE; therefore the remaining angle BAC is equal [d] to the remaining angle EDF: Wherefore the triangle d 32. 1.
ABC is equiangular to the triangle DEF, and it is inſcribed in the circle ABC. Which was to be done.

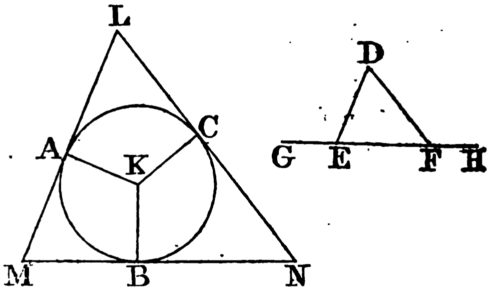


PROP. III. PROB.

ABOUT a given circle to deſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to deſcribe a triangle about the circle ABC equiangular to the triangle DEF.

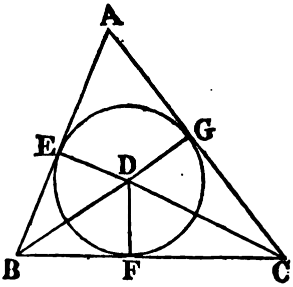
Produce EF both ways to the points G, H, and find the
centre K of the circle ABC, and from it draw any ſtraight
a 23. 1. line KB; at the point K in the ſtraight line KB, make [a] the
angle BKA equal to the angle DEG, and the angle BKC
equal to the angle DFH; and through the points A, B, C,
b 17. 3. draw the ſtraight lines LAM, MBN, NCL touching [b] the
circle ABC; Therefore, becauſe LM, MN, NL touch the
circle ABC in the points A, B, C, to which from the centre
are drawn KA, KB, KC, the angles at the points A, B, C, are
c 18. 3. right [c] angles. And becauſe the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can
be divided into two triangles, and becauſe two of them, KAM,
KBM are right angles, the other two AKB, AMB are equal to two right angles: But the angles DEG,
d 13. 1. DEF are likewise equal [d] to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal
to DEG; wherefore the remaining angle AMB is equal to
the remaining angle DEF: In like manner, the angle LNM
may be demonſtrated to be equal to DFE; and therefore the
e 32. 1. remaining angle MLN is equal [e] to the remaining angle EDF:
Wherefore the triangle LMN is equiangular to the triangle
DEF: And it is deſcribed about the circle ABC. Which
was to be done.



PROP. IV. PROB.

inſcribe a circle in a given triangle.

the given triangle be ABC; it is required to inſcribe
e in ABC.
ct [a] the angles ABC, BCA by the ſtraight lines BD, a 9. 1.
eeting one another in the point D, from which draw [b] b 12. 1.
DF, DG perpendiculars
B, BC, CA: And be-
the angle EBD is equal
angle FBD, the angle
being biſected by BD;
ecauſe the right angle
is equal to the right angle
the two triangles EBD,
have two angles of the
ual to two angles of the
and the ſide BD, which
oſite to one of the equal



in each, is common to both; therefore their other ſides
qual [c]; wherefore DE is equal to DF. For the ſame c 26. 1.
DG is equal to DF; therefore the three ſtraight lines
DF, DG are equal to one another, and the circle de-
d from the centre D, at the diſtance of any of them, will
rough the extremities of the other two, and will touch the
t lines AB, BC, CA, becauſe the angles at the points
G are right angles, and the ſtraight line which is drawn
the extremity of a diameter at right angles to it, touches
circle: Therefore the ſtraight lines AB, BC, CA, do d 16. 3.
f them touch the circle, and the circle EFG is inſcribed
triangle ABC. Which was to be done.

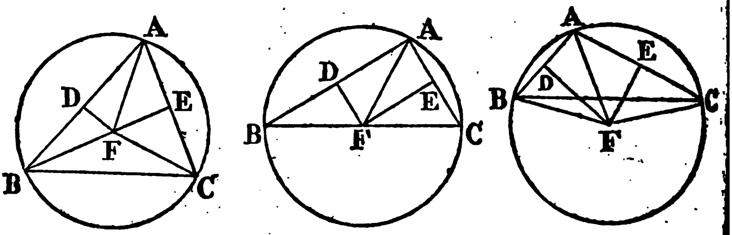
PROP. V. PROB.

To deſcribe a circle about a given triangle.

Let the given triangle be ABC; it is required to deſcribe a circle about ABC.

a 10. 1. Biſect[a] AB, AC in the points D, E, and from theſe points

b 11. 1. draw DF, EF at right angles[b] to AB, AC; DF, EF produced



meet one another; for, if they do not meet, they are paralle wherefore AB, AC, which are at right angles to them, an parallel, which is abſurd: Let them meet in F, and join FA also, if the point F be not in BC, join BF, CF: then, becauſ AD is equal to DB, and DF common, and at right angles t

c 4. 1. AB, the baſe AF is equal[c] to the baſe FB. In like manne it may be ſhown that CF is equal to FA; and therefore B is equal to FC; and FA, FB, FC are equal to one another wherefore the circle deſcribed from the centre F, at the diſtance of one of them, ſhall paſs through the extremities of th other two, and be deſcribed about the triangle ABC, whic was to be done.

Cor. And it is manifeſt, that when the centre of the circl falls within the triangle, each of its angles is leſs than a righ angle, each of them being in a ſegment greater than a ſemicircle; but, when the centre is in one of the ſides of the triangle, the angle oppoſite to this ſide, being in a ſemicircle, i a right angle; and, if the centre falls without the triangle, th angle oppoſite to the ſide beyond which it is, being in a ſegment leſs than a ſemicircle, is greater than a right angle

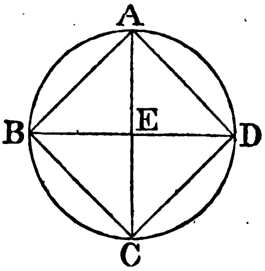
Wherefor

Wherefore, if the given triangle be acute angled, the centre of the circle falls within it; if it be a right angled triangle, the centre is in the ſide oppoſite to the right angle; and, if it be an obtuſe angled triangle, the centre falls without the triangle, beyond the ſide oppoſite to the obtuſe angle.

PROP. VI. PROB.

TO inſcribe a ſquare in a given circle.

Let ABCD be the given circle; it is required to inſcribe a ſquare in ABCD.



Draw the diameters AC, BD at right angles to one another, and join AB, BC, CD, DA; becauſe BE is equal to ED, E being the centre, and becauſe EA is at right angles to BD, and common to the triangles ABE, ADE; the baſe BA is equal [a] to the baſe AD; and, for the ſame reaſon, BC, CD are each of them equal to BA or AD; therefore the quadrilateral figure ABCD is equilateral. It is alſo rectangular; for the ſtraight line BD, being the diameter of the circle ABCD, BAD is a ſemicircle; wherefore the angle BAD is a right [b] angle; for the ſame reaſon each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been ſhown to be equilateral; therefore it is a ſquare; and it is inſcribed in the circle ABCD. Which was to be done.

a 4. 1.

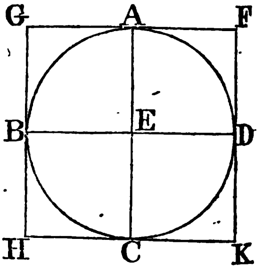
b 31. 3.

PROP. VII. PROB.

TO deſcribe a ſquare about a given circle.

Let ABCD be the given circle; it is required to deſcribe a ſquare about it.

Draw two diameters AC, BD of the circle ABCD, at righ angles to one another, and through the points A, B, C, I a 17. 3. draw [a] FG, GH, HK, KF touching the circle; and becauſ FG touches the circle ABCD, and EA is drawn from th centre E to the point of contact A, the angles at A are righ b 18. 3. [b] angles; for the ſame reaſon, the angles at the points B, C, D are right angles; and becauſe the angle AEB is a right angle, as likewiſe is EBG, GH c 28. 1. is parallel [c] to AC; for the ſame reaſon, AC is parallel to FK, and in like manner GF, HK may each of them be demonſtrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms; and GF d 34. 1. is therefore equal [d] to HK, and GH to FK; and becauſe AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK: GH, FK are each of them equal to GF or HK; therefore the quadrilateral figure FGHK is equilateral. It is alſo rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB [d] is likewiſe a right angle: In the ſame manner, it may be ſhown that the angles at H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular; and it was demonſtrated to be equilateral; therefore it is a ſquare; and it is deſcribed about the circle ABCD. Which was to be done.

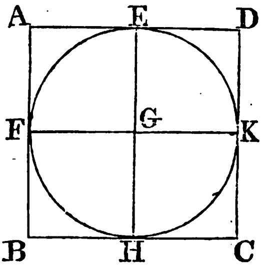


PROP. VIII. PROB.

To inſcribe a circle in a given ſquare.

Let ABCD be the given ſquare; it is required to inſcribe a circle in ABCD.

a 10. 1. Biſect [a] each of the ſides AB, AD, in the points F, E, and b 31. 1. through E draw [b] EH parallel to AB or DC, and through F draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, c 34. 1. and their oppoſite ſides are equal [c]; and becauſe AD is equal to AB, and that AE is the half of AD, and AF the half of AB, AE is equal to AF; wherefore the ſides oppoſite to theſe are equal, viz. FG to GE; in the ſame manner, it may be demonſtrated, that GH, GK are each of them equal to FG or GE; therefore the four ſtraight lines GE, GF, GH, GK, are equal to one another; and the circle deſcribed from the centre G, at the diſtance of one of them, ſhall paſs through the extremities of the other three; and ſhall alſo touch the ſtraight lines AB, BC, CD, DA, becauſe the angles at the points E, F, H, K are right[d] angles, and becauſe the ſtraight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle[e]; therefore each of the ſtraight lines AB, BC, CD, DA touches the circle, which therefore is inſcribed in the ſquare ABCD. Which was to be done.



d 29. 1.

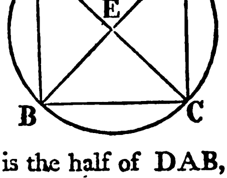
e 16. 3.

PROP. IX. PROB.

To deſcribe a circle about a given ſquare.

Let ABCD be the given ſquare; it is required to deſcribe a circle about it.

Join AC, BD, cutting one another in E; and becauſe DA is equal to AB, and AC common to the triangles DAC, BAC, the two ſides DA, AC are equal to the two BA, AC, and the baſe DC is equal to the baſe BC; wherefore the angle DAC is equal[a] to the angle BAC, and the angle DAB is biſected by the ſtraight line AC: In the ſame manner, it may be demonſtrated, that the angles ABC, BCD, CDA are ſeverally biſected by the ſtraight lines BD, AC; therefore, becauſe the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and



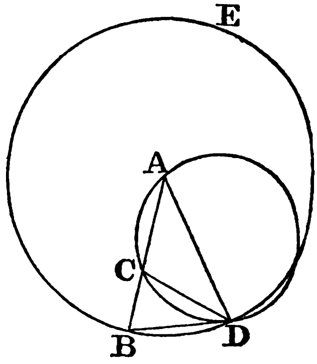
a 8. 1.

and EBA the half of ABC; the angle EAB is equal to the angle EBA; wherefore the ſide EA is equal b to the ſide EB: In the ſame manner, it may be demonſtrated, that the ſtraight lines EC, ED are each of them equal to EA or EB; therefore the four ſtraight lines EA, EB, EC, ED are equal to one another; and the circle deſcribed from the centre E, at the diſtance of one of them, ſhall paſs through the extremities of the other three, and be deſcribed about the ſquare ABCD. Which was to be done. b 6. 1.

PROP. X. PROB.

TO deſcribe an iſoſceles triangle, having each of the angles at the baſe double of the third angle.

a 11. 2. Take any ſtraight line AB, and divide a it in the point C, ſo that the rectangle AB, BC may be equal to the ſquare of CA; and from the centre A, at the diſtance AB, deſcribe the circle BDE, in which place b the ſtraight line BD equal to AC, which is not greater than the diameter of the circle BDE; join DA, DC, and about the triangle ADC deſcribe c the circle ACD; the triangle ABD is ſuch as is required, that is, each of the angles ABD, ADB is double of the angle BAD. b 1. 4. c 5. 4.



Becauſe the rectangle AB, BC is equal to the ſquare of AC, and that AC is equal to BD, the rectangle AB, BC is equal to the ſquare of BD; and becauſe from the point B without the circle ACD two ſtraight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC contained by the whole of the cutting line, and the part of it without the circle, is equal to the ſquare of BD which meets it; the ſtraight line BD touches d the circle ACD; and becauſe BD touches the circle, and DC is d 37. 3.

drawn from the point of contact D, the angle BDC is equal [e]
to the angle DAC in the alternate ſegment of the circle; to
each of theſe add the angle CDA; therefore the whole angle 32. 3.
BDA is equal to the two angles CDA, DAC; but the exterior
angle BCD is equal [f] to the angles CDA, DAC; 32. 1.
therefore alſo BDA is equal to BCD; but BDA is equal [g] g 5. 1.
to CBD, becauſe the ſide AD is equal to the ſide AB;
therefore CBD, or DBA is equal to BCD; and conſequently
the three angles BDA, DBA, BCD, are equal to
one another: And becauſe the angle DBC is equal to the angle
BCD, the ſide BD is equal [h] to the ſide DC; but BD was h 6. 1.
made equal to CA; therefore alſo CA is equal to CD, and
the angle CDA equal [g] to the angle DAC; therefore the
angles CDA, DAC together, are double of the angle DAC:
but BCD is equal to the angles CDA, DAC; therefore alſo
BCD is double of DAC; and BCD is equal to each of the
angles BDA, DBA, each therefore of the angles BDA,
DBA is double of the angle DAB; wherefore an iſoſceles
triangle ABD is deſcribed, having each of the angles at the
baſe double of the third angle. Which was to be done.

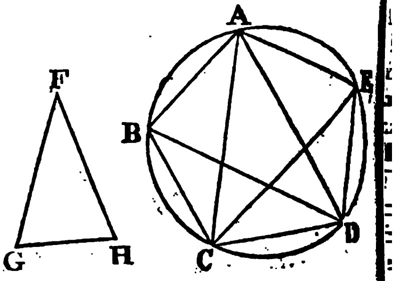
PROP. XI. PROB.

TO inſcribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle, it is required to inſcribe an equilateral and equiangular pentagon in the circle ABCDE.

Deſcribe [a] an iſoſceles triangle FGH, having each of the a 10. 4.
angles at G, H, double of the angle at F; and in the circle
ABCDE inſcribe [b] the triangle ACD equiangular to the b 2. 4.

triangle FGH, ſo that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the angle CAD. c 9. 1. Biſect c the angles ACD, CDA by the ſtraight lines CE, DB; and join AB, BC, DE, EA. ABCDE is the pentagon required.



Becauſe angles the ACD, CDA are each of them double of CAD, and are biſected by the ſtraight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another;
d 26. 3. but equal angles ſtand upon equal d circumferences; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another: and equal circumferences are ſubtended by
e 29. 3. equal e ſtraight lines; therefore the five ſtraight lines AB, BC, CD, DE, EA are equal to one another. Wherefore the pentagon ABCDE is equilateral. It is alſo equiangular; becauſe the circumference AB is equal to the circumference DE: if to each be added BCD, the whole ABCD is equal to the whole EDCB: and the angle AED ſtands on the circumference ABCD, and the angle BAE on the circumference
f 27. 3. EDCB; therefore the angle BAE is equal f to the angle AED: for the ſame reaſon, each of the angles ABC, BCD, CDE is equal to the angle BAE, or AED: therefore the pentagon ABCDE is equiangular; and it has been ſhown that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inſcribed. Which was to be done.

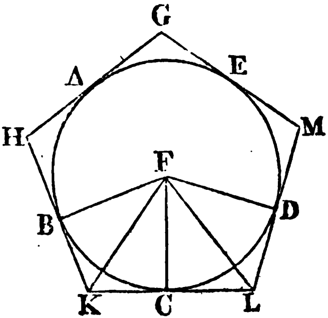
PROP. XII. PROB.

TO deſcribe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle, it is required to deſcribe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon, inſcribed in the circle, by the laſt propoſition, be in the points A, B, C, D, E, ſo that the circumferences AB, BC, CD, DE, EA are equal[a]; and through the points A, B, C, D, E draw GH, HK, KL, LM, MG, touching[b] the circle: take the centre F, and join FB, FK, FC, FL, FD: and becauſe the ſtraight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular[c] to KL; therefore each of the angles at C is a right angle: for the ſame reaſon, the angles at the points B, D are right angles; and becauſe FCK is a right angle, the ſquare of FK is equal[d] to the ſquares of FC, CK. For the ſame reaſon, the ſquare of FK is equal to the ſquares of FB, BK: therefore the ſquares of FC, CK are equal to the ſquares of FB, BK, of which the ſquare of FC is equal to the ſquare of FB; the remaining ſquare of CK is therefore equal to the remaining ſquare of BK, and the ſtraight line CK equal to BK: and becauſe FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK; and the baſe BK is equal to the baſe KC; therefore the angle BFK is equal[e] to the angle KFC, and the angle BKF to FKC; wherefore the angle BFC is double of the angle KFC, and BKC double of FKC: for the ſame reaſon, the angle CFD is double of the angle CFL, and CLD double of CLF: and becauſe the circumference BC is equal to the circumference CD, the angle BFC is equal[f] to the angle CFD; and BFC is double of the angle KFC, and CFD double of CFL; therefore the angle KFC is equal to the angle CFL; and the right angle FCK is equal to the right angle FCL: therefore in the two triangles FKC, FLC, there are two angles of one equal to two angles of the other, each to each, and the ſide FC, which is adjacent to the equal angles

a 11. 4.
b 17. 3.
c 18. 3.
d 47. 1.
e 8. 1.
f 27. 3.



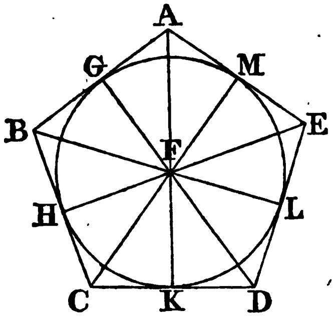
in each, is common to both; therefore the other ſides ſhall be
g 26. 1. equal [g] to the other ſides, and the third angle to the third angle: therefore the ſtraight line KC is equal to CL, and the angle FKC to the angle FLC: and becauſe KC is equal to CL, KL is double of KC: in the ſame manner, it may be ſhown that HK is double of BK: and becauſe BK is equal to KC, as was demonſtrated, and that KL is double of KC, and HK double of BK, HK ſhall be equal to KL: in like manner, it may be ſhown that GH, GM, ML are each of them equal to HK or KL: therefore the pentagon GHKLM is equilateral. It is alſo equiangular; for, ſince the angle FKC is equal to the angle FLC, and the angle HKL double of the angle FKC, and KLM double of FLC, as was before demonſtrated, the angle HKL is equal to KLM: and in like manner it may be ſhown, that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM: therefore the five angles GHK, HKL, KLM, LMG, MGH being equal to one another, the pentagon GHKLM is equiangular: and it is equilateral, as was demonſtrated; and it is deſcribed about the circle ABCDE. Which was to be done.

PROP. XIII. PROB.

To inſcribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inſcribe a circle in the pentagon ABCDE.

a 9. 1. Biſect [a] the angles BCD, CDE by the ſtraight lines CF, DF, and from the point F, in which they meet, draw the ſtraight lines FB, FA, FE: therefore, ſince BC is equal to CD, and CF common to the triangles BCF, DCF, the two ſides BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF; therefore the baſe BF is
b 4. 1. equal [b] to the baſe FD, and the other angles to the other angles, to which the equal ſides are oppoſite; therefore the angle CBF is equal to the angle CDF: and becauſe the angle



angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF; CBA is alſo double of the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is biſected by the ſtraight line BF: In the ſame manner, it may be demonſtrated, that the angles BAE, AED, are biſected by the ſtraight lines AF, FE: from the point F draw [c] FG, FH, FK, FL, FM perpendiculars to the ſtraight lines AB, BC, CD, DE, EA: and becauſe the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC; in the triangles FHC, FKC there are two angles of one equal to two angles of the other, and the ſide FC, which is oppoſite to one of the equal angles in each, is common to both; therefore the others ſides ſhall be equal [d], each to each; wherefore the perpendicular FH is equal to the perpendicular FK: in the ſame manner it may be demonſtrated, that FL, FM, FG are each of them equal to FH or FK: therefore the five ſtraight lines FG, FH, FK, FL, FM are equal to one another: wherefore the circle deſcribed from the centre F, at the diſtance of one of theſe five, ſhall paſs through the extremities of the other four, and touch the ſtraight lines AB, BC, CD, DE, EA, becauſe that the angles at the points G, H, K, L, M are right angles, and that a ſtraight line drawn from the extremity of the diameter of a circle at right angles to it, touches [e] the circle: therefore each of the ſtraight lines AB, BC, CD, DE, EA touches the circle; wherefore it is inſcribed in the pentagon ABCDE. Which was to be done.

c 12. 1.

d 26. 1.

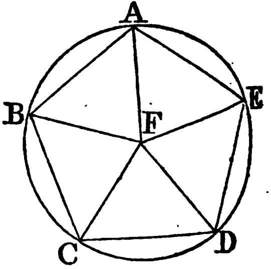
PROP.

PROP. XIV. PROB.

TO deſcribe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to deſcribe a circle about it.

a 9. 1. Biſect [a] the angles BCD, CDE by the ſtraight lines CF, FD, and from the point F, in which they meet, draw the ſtraight lines FB, FA, FE to the points B, A, E. It may be demonſtrated, in the ſame manner as in the preceding propoſition, that the angles CBA, BAE, AED are biſected by the ſtraight lines FB, FA, FE: and becauſe the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle



b 6. 1. FCD is equal to FDC; wherefore the ſide CF is equal [b] to the ſide FD: In like manner it may be demonſtrated, that FB, FA, FE are each of them equal to FC or FD: therefore the five ſtraight lines FA, FB, FC, FD, FE are equal to one another; and the circle deſcribed from the centre F, at the diſtance of one of them, ſhall paſs through the extremities of the other four, and be deſcribed about the equilateral and equiangular pentagon ABCDE. Which was to be done.

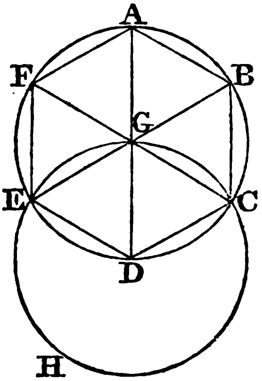
PROP. XV. PROB.

TO inſcribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inſcribe an equilateral and equiangular hexagon in it.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D as a centre, at the diſtance DG, deſcribe the circle EGCH, join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA: the hexagon ABCDEF is equilateral and equiangular.

Becauſe G is the centre of the circle ABCDEF, GE is equal to GD: and becauſe D is the centre of the circle EGCH, DE is equal to DG; wherefore GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG are equal to one another [a]; and the three angles of a triangle are equal [b] to two right angles; therefore the angle EGD is the third part of two right angles: In the ſame manner it may be demonſtrated that the angle DGC is alſo the third part of two right angles: and becauſe the ſtraight line GC makes with EB the adjacent angles EGC, CGB equal [c] to two right angles; the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB, are equal to one another: and to theſe are equal [d] the vertical oppoſite angles BGA, AGF, FGE; therefore the ſix angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another. But equal angles ſtand upon equal [e] circumferences; therefore the ſix circumferences AB, BC, CD, DE, EF, FA are equal to one another: and equal circumferences are ſubtended by equal [f] ſtraight lines; therefore the ſix ſtraight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is alſo equiangular; for, ſince the circumference AF is equal to ED, to each of theſe add the circumference ABCD; therefore the whole circumference FABCD ſhall be equal to the whole EDCBA: and the angle FED ſtands upon the circumference



[a] Cor. 5. 1.
[b] 32. 1.
[c] 13. 1.
[d] 15 3.
[e] 26. 3.
[f] 29. 3.

cumference FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal to FED: in the ſame manner it may be demonſtrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED; therefore the hexagon is equiangular; and it is equilateral, as was ſhown; and it is inſcribed in the given circle ABCDEF. Which was to be done.

Cor. From this it is manifeſt, that the ſide of the hexagon is equal to the ſtraight line from the centre, that is, to the ſemidiameter of the circle.

And if through the points A, B, C, D, E, F there be drawn ſtraight lines touching the circle, an equilateral and equiangular hexagon ſhall be deſcribed about it, which may be demonſtrated from what has been ſaid of the pentagon; and likewiſe a circle may be inſcribed in a given equilateral and equiangular hexagon, and circumſcribed about it, by a method like to that uſed for the pentagon.

PROP. XVI. PROB.

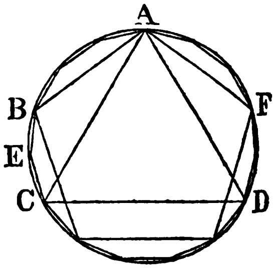
To inſcribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle; it is required to inſcribe an equilateral and equiangular quindecagon in the circle ABCD.

a 2. 4. Let AC be the ſide of an equilateral triangle inſcribed [a] in the circle, and AB the ſide of an equilateral and equiangular pen-

b 11. 4. tagon inſcribed [b] in the ſame; therefore, of ſuch equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, being the third part of the whole, contains five; and the circumference AB, which is the fifth part of the whole, contains three; therefore

c 30. 3. BC their difference contains two of the ſame parts: biſect [c]



ı E; therefore BE, EC are, each of them, the fifteenth of the whole circumference ABCD: therefore if the ıt lines BE, EC be drawn, and ſtraight lines equal to be placed [d] around in the whole circle, an equilateral d. 1. 4. quiangular quindecagon ſhall be inſcribed in it. Which ɔ be done.

ıd in the ſame manner as was done in the pentagon, if gh the points of diviſion made by inſcribing the quindecagon, ſtraight lines be drawn touching the circle, an equilateral and equiangular quindecagon ſhall be deſcribed about it: likewiſe, as in the pentagon, a circle may be inſcribed in :n equilateral and equiangular quindecagon, and circumſcribed about it.

ELEMENTS

OF

GEOMETRY.

BOOK V.

DEFINITIONS.

I.

A Leſs magnitude is ſaid to be a part of a greater magnitude, when the leſs meaſures the greater, that is, when the leſs is contained a certain number of times exactly in the greater.

II.

A greater magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs, that is, when the greater contains the leſs a certain number of times exactly.

III.

Ratio is a mutual relation of two magnitudes of the ſame kind to one another, in reſpect of quantity.

IV.

Magnitudes are ſaid to be of the ſame kind, when the leſs can be multiplied ſo as to exceed the greater; and it is only ſuch magnitudes that are ſaid to have a ratio to one another.

V.

See N. If there be four magnitudes, and if any equimultiples whatſoever be taken of the firſt and third, and any equimultiples whatſoever of the ſecond and fourth, and if, according as the multiple of the firſt is greater than the multiple of the ſecond, equal to it, or leſs, the multiple of the third is alſo greater than the multiple of the fourth, equal to it, or leſs; then the firſt of the magnitudes is ſaid to have to the ſecond the ſame ratio that the third has to the fourth.

VI.

Magnitudes are ſaid to be proportionals, when the firſt has the ſame ratio to the ſecond that the third has to the fourth, and the third to the fourth the ſame ratio which the fifth has to the ſixth, and ſo on, whatever be their number.

When four magnitudes, A, B, C, D are proportionals, it is uſual to ſay that A is to B as C to D, and to write them thus, A : B : : C : D, or thus, A : B = C : D.

VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition) the multiple of the firſt is greater than that of the ſecond, but the multiple of the third is not greater than the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is ſaid to have to the fourth a leſs ratio than the firſt has to the ſecond.

VIII.

When there is any number of magnitudes greater than two, of which the firſt has to the ſecond the ſame ratio that the ſecond has to the third, and the ſecond to the third the ſame ratio which the third has to the fourth, and ſo on, the magnitudes are ſaid to be continual proportionals.

IX.

When three magnitudes are continual proportionals, the ſecond is ſaid to be a mean proportional between the other two.

X.

When there is any number of magnitudes of the ſame kind, the firſt is ſaid to have to the laſt of them the ratio compounded of the ratio which the firſt has to the ſecond, and of the ratio which the ſecond has to the third, and of the ratio which the third has to the fourth; and ſo on unto the laſt magnitude.

For example, if A, B, C, D be four magnitudes of the ſame kind, the firſt A is ſaid to have to the laſt D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is ſaid to be compounded of the ratios of A to B, B to C, and C to D.

And if A : B : : E : F; and B : C : : G : H, and C : D : : K : L, then, ſince by this definition A has to D the ratio compounded of the ratios of A to B, B to C, C to D; A may alſo be ſaid to have to D the ratio compounded of the ratios which are the ſame with the ratios of E to F, G to H, and K to L.

In like manner, the ſame things being ſuppoſed, if M has to N the ſame ratio which A has to D, then, for ſhortneſs ſake, M is ſaid to have to N a ratio compounded of the ſame ratios which compound the ratio of A to D; that is, a ratio compounded of the ratios of E to F, G to H, and K to L.

XI.

A ratio which is compounded of two equal ratios is ſaid to be duplicate of either of theſe ratios.

Cor. Hence, if the three magnitudes A, B, and C are continual proportionals, the ratio of A to C is duplicate of that of A to B, or of B to C: For by the laſt definition, the ratio of A to C is compounded of the ratios of A to B, and of B to C; but the ratio of A to B is equal to the ratio of B to C, becauſe A, B, C are continual proportionals; therefore the ratio of A to C, by this definition, is duplicate of the ratio of A to B, or of B to C.

XII.

A ratio which is compounded of three equal ratios is ſaid to be triplicate of any one of theſe ratios; and a ratio which is compounded of four equal ratios is ſaid to be quadruplicate of any one of theſe ratios; and ſo on, according to the number of equal ratios.

Cor. If four magnitudes, A, B, C, D be continual proportionals, the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D. For the ratio of A to D is compounded of the three ratios of A to B, B to C, C to D; and theſe three ratios are equal to one another, becauſe A, B, C, D are continual proportionals, therefore the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D.

XIII.

In proportionals, the antecedent terms are called homologous to one another, as alſo the conſequents to one another.

Geometers make uſe of the following technical words to ſignify certain ways of changing either the order or magnitude of proportionals, ſo as that they continue ſtill to be proportionals.

XIV.

Permutando, or alternando, by permutation, or alternately; this word is uſed when there are four proportionals, and it is inferred, that the firſt has the ſame ratio to the third, which the ſecond has to the fourth; or that the firſt is to the third, as the ſecond to the fourth: As is ſhewn in the 16th prop. of this 5th book.

XV.

Invertendo, by inverſion: When there are four proportionals, and it is inferred, that the ſecond is to the firſt, as the fourth to the third. Prop. A. book 5.

XVI.

Componendo, by compoſition: When there are four proportionals, and it is inferred, that the firſt, together with the ſecond, is to the ſecond, as the third, together with the fourth, is to the fourth. 18th prop. book 5.

XVII.

Dividendo, by diviſion: When there are four proportionals, and it is inferred, that the exceſs of the firſt above the ſecond, is to the ſecond, as the exceſs of the third above the fourth, is to the fourth. 17th prop. book 5.

XVIII.

Convertendo, by converſion: When there are four proportionals, and it is inferred, that the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth. Prop. D. book 5.

XIX.

Ex aequali (ſc. diſtantia), or ex aequo, from equality of diſtance; when there is any number of magnitudes more than two, and as many others, ſo that they are proportionals when taken two and two of each rank, and it is inferred, that the firſt is to the laſt of the firſt rank of magnitudes, as the firſt is to the laſt of the others: Of this there are the two following kinds, which ariſe from the different order in which the magnitudes are taken two and two.

XX.

Ex aequali, from equality; this term is uſed ſimply by itſelf, when the firſt magnitude is to the ſecond of the firſt rank, as the firſt to the ſecond of the other rank; and as the ſecond is to the third of the firſt rank, ſo is the ſecond to the third of the other; and ſo on in order, and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonſtrated in the 22d prop. book 5.

XXI.

Ex aequali, in proportione perturbata, ſeu inordinata; from equality, in perturbate or diſorderly proportion; this term is uſed when the firſt magnitude is to the ſecond of the

the firſt rank, as the laſt but one is to the laſt of the f
rank; and as the ſecond is to the third of the firſt ra
is the laſt but two to the laſt but one of the ſecond
and as the third is to the fourth of the firſt rank, ſo
third from the laſt to the laſt but two of the ſecond
and ſo on in a croſs order; and the inference is as i
19th definition. It is demonſtrated in the 23d pr
book 5.

IN the demonſtrations of this book there are c
ſigns or *characters* which it has been found conveni
employ.

1. The letters A, B, C, &c. are uſed to denote magn
of any kind. The letters *m*, *n*, *p*, *q*, are uſed to d
numbers only.

2. The ſign + (plus), written between two letter
denote magnitudes or numbers, ſignifies the ſum of thoſe
nitudes or numbers. Thus, A + B is the ſum of th
magnitudes denoted by the letters A and B; *m* + *n*
ſum of the numbers denoted by *m* and *n*.

3. The ſign — (minus), written between two lette
nifies the exceſs of the magnitude denoted by the f
theſe letters, which is ſuppoſed the greateſt, above that
is denoted by the other. Thus, A—B ſignifies the ex
the magnitude A above the magnitude B.

4. When a number, or a letter denoting a number, i
ten cloſe to another letter denoting a magnitude of any
it ſignifies that the magnitude is multiplied by the n
Thus, 3A ſignifies three times A; *m*B *m* times B,
multiple of B by *m*. When the number is intended t
tiply two or more magnitudes that follow, it is writte
$m.\overline{A+B}$. which ſignifies the ſum of A and B taken *m*
$m.\overline{A-B}$ is *m* times the exceſs of A above B.

Alſo, when two letters that denote numbers are v
cloſe to one another, they denote the product of thoſe
bers, when multiplied into one another. Thus, *mn*
product of *m* into *n*.

5. The ſign $=$ ſignifies the equality of the magnitudes denoted by the letters that ſtand on the oppoſite ſides of it; $A=B$ ſignifies that A is equal to B; $A+B=C-D$ ſignifies that the ſum of A and B is equal to the exceſs of C above D.

6. The ſign $>$ is uſed to ſignify the inequality of the magnitudes between which it is placed, and that the magnitude to which the opening of the lines is turned is greater than the other. Thus, $A>B$ ſignifies that A is greater than B; $A<B$ ſignifies that A is leſs than B. This ſign is uſed, too, to expreſs the inequality of ratios, thus, $A:B>C:D$ ſignifies that the ratio of A to B is greater than that of C to D; and $A:B<C:D$ ſignifies that the ratio of A to B is leſs than that of C to D.

AXIOMS.

I.

EQUIMULTIPLES of the ſame, or of equal magnitudes, are equal to one another.

II.

Thoſe magnitudes of which the ſame, or equal magnitudes, are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the ſame multiple of a leſs.

IV.

That magnitude of which a multiple is greater than the ſame multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

If any number of magnitudes be equimultiples of as many others, each of each, what multiple ſoever any one of the firſt is of its part, the ſame multiple is the ſum of all the firſt of the ſum of all the reſt.

Let any number of magnitudes A, B, and C be equimultiples of as many others D, E, and F, each of each; A+B+C is the ſame multiple of D+E+F, that A is of D.

Let A contain D, B contain E, and E contain F, each any number of times, as, for inſtance, three times. Then, becauſe A contains E three times A = D+D+D.

For the ſame reaſon, B = E+E+E;
And alſo, C = F+F+F.

a Ax. 2. 1. Therefore, adding equals to equals, [a] A+B+C is equal to D+E+F taken three times. In the ſame manner, if A, B, and C were each any other equimultiple of D, E, and F, it would be ſhewn that A+B+C was the ſame multiple of D+E+F. Therefore, &c. Q. E. D.

Cor. Hence, if m be any number, $mD+mE+mF = m.\overline{D+E+F}$. For mD, mE, and mF are multiples of D, E, and F by m, therefore their ſum is alſo a multiple of D+E+F by m.

PROP. II. THEOR.

If to a multiple of a magnitude by any number a multiple of the ſame magnitude by any number be added, the ſum will be the ſame multiple of that magnitude that the ſum of the two numbers is of unity.

Let $A=mC$, and $B=nC$; $A+B=\overline{m+n}.C$.

For, ſince $A=mC$, A=C+C+C+ &c. C being repeated m times. For the ſame reaſon, B=C+C+ &c, C being repeated n times. Therefore, adding equals to equals, A+B is equal to C taken $m+n$ times; that is, $A+B=\overline{m+n}.C$. Therefore A+B contains C as oft as there are units in $m+n$. Q. E. D.

Cor. 1. In the ſame way, if there be any number of multiples whatſoever, as $A=mE$, $B=nE$, $C=pE$, it is ſhewn, that $A+B+C=\overline{m+n+p}.E$.

Cor. 2. Hence alſo, ſince $A+B+C=\overline{m+n+p}.E$, and ſince $A=mE$, $B=nE$, and $C=pE$, $mE+nE+pE=\overline{m+n+p}.E$.

PROP. III. THEOR.

If the firſt of three magnitudes contain the ſecond as oft as there are units in a certain number, and if the ſecond contain the third alſo as often as there are units in a certain number, the firſt will contain the third as oft as there are units in the product of theſe two numbers.

Let $A=mB$, and $B=nC$; then $A=mnC$.

Since $B=nC$, $mB=nC+nC+$ &c. repeated m times. But $nC+nC+$ &c. repeated m times, is equal[a] to C multiplied by $n+n+$ &c. n being repeated m times. And n repeated m times is mn; therefore $mB=mnC$. But by hypotheſis, $A=mB$; therefore $A=mnC$.

[a] 2. 5.

PROP. IV. THEOR.

If the firſt of four magnitudes has the ſame ratio to the ſecond which the third has to the fourth, and if any equimultiples whatever be taken of the firſt and third, and any whatever of the ſecond and fourth; the multiple of the firſt ſhall have the ſame ratio to the multiple of the ſecond that the multiple of the third has to the multiple of the fourth.

Let $A : B :: C : D$, and let m and n be any two numbers; $mA : nB :: mC : nD$.

Take of mA and mC equimultiples by any number p, and of nB and nD equimultiples by any number q. Then the equimultiples of mA and mC by p, are equimultiples alſo of A and C, for they contain A and C as oft as there are units a 3. 5. in pm[a], and are equal to pmA and pmC. For the ſame reaſon the multiples of nB and nD by q, are qnB, qnD. Since therefore $A : B :: C : D$, and of A and C there are taken any equimultiples, viz. pmA and pmC, and of B and D, any equimultiples qnB, qnD, if pmA be greater than qnB, pmC b def. 5. 5. muſt be greater than qnD[b]; if equal, equal; and if leſs, leſs. But pmA, pmC are alſo equimultiples of mA and mC, and qnB, qnD are equimultiples of nB and nD, therefore[b], $mA : nB :: mC : nD$. Therefore, &c. Q. E. D.

Cor. In the ſame manner it may be demonſtrated, that if $A : B :: C : D$, and of A and C equimultiples be taken by any number m, viz. mA and mC, $mA : B :: mC : D$. This may alſo be conſidered as included in the propoſition, and as being the caſe when $n = 1$.

PROP. V. THEOR.

If one magnitude be the ſame multiple of another, which a magnitude taken from the firſt is of a magnitude taken from the other; the remainder ſhall be the ſame multiple of the remainder, that the whole is of the whole.

Let mA and mB be any equimultiples of the two magnitudes A and B, of which A is greater than B; $mA - mB$ is the ſame multiple of $A - B$ that mA is of A, that is, $mA - mB = m.\overline{A - B}$.

Let D be the exceſs of A above B, then $A - B = D$, and a 1. 5. adding B to both $A = D + B$. Therefore[a] $mA = mD + mB$; take mB from both, and $mA - mB = mD$; but $D = A - B$, therefore $mA - mB = m.\overline{A - B}$. Therefore, &c. Q. E. D.

PROP. VI. THEOR.

IF from a multiple of a magnitude by any number a multiple of the ſame magnitude by a leſs number be taken away, the remainder will be the ſame multiple of that magnitude that the difference of the numbers is of unity.

Let mA and nA be multiples of the magnitude A by the numbers m and n, and let m be greater than n; $mA-nA$ contains A as oft as $m-n$ contains unity, or $mA-nA=\overline{m-n}.A$.

Let $m-n=q$; then $m=n+q$. Therefore [a] $mA=nA+qA$; take nA from both, and $mA-nA=qA$. Therefore $mA-nA$ contains A as oft as there are units in q, that is, in $m-n$, or $mA-nA=\overline{m-n}.A$. Therefore, &c. Q. E. D. a 2. 5.

Cor. When the difference of the two numbers is equal to unity, or $m-n=1$, then $mA-nA=A$.

PROP. A. THEOR.

IF four magnitudes be proportionals, they are proportionals alſo when taken inverſely.

If A : B : : C : D, then alſo B : A : : D : C.

Let mA and mC be any equimultiples of A and C; nB and nD any equimultiples of B and D. Then, becauſe A : B : : C : D, if mA be leſs than nB, mC will be leſs than nD [a], that is, if nB be greater than mA, nD will be greater than mC. For the ſame reaſon, if $nB=mA$, $nD=mC$, and if $nB<mA$, $nD<mC$. But nB, nD are any equimultiples of B and D, and mA, mC any equimultiples of A and C, therefore [a], B : A : : D : C. Therefore, &c. Q. E. D. a def. 5. 5.

PROP. B. THEOR.

IF the firſt be the ſame multiple of the ſecond, or the ſame part of it, that the third is of the fourth; the firſt is to the ſecond as the third to the fourth.

Firſt, if mA, mB be equimultiples of the magnitudes A and B, $mA : A :: mB : B$.

Take of mA and mB equimultiples by any number n; and of A and B equimultiples by any number p; theſe will be a 3. 5. nmA [a], pA, nmB [a], pB. Now, if nmA be greater than pA, nm is alſo greater than p; and if nm is greater than p, nmB is greater than pB; therefore, when nmA is greater than pA, nmB is greater than pB. In the ſame manner, if $nmA = pA$, $nmB = pB$, and if $nmA < pA$, $nmB < pB$. Now, nmA, nmB are any equimultiples of mA and mB; and pA, pB are b def. 5. 5. any equimultiples of A and B, therefore $mA : A :: mB : B$ [b].

Next, Let C be the ſame part of A that D is of B; then A is the ſame multiple of C that B is of D; and therefore, c A. 5. as has been demonſtrated, $A : C :: B : D$, and inverſely [c] $C : A :: D : B$. Therefore, &c. Q. E. D.

PROP. VI. THEOR.

IF the firſt be to the ſecond as the third to the fourth; and if the firſt be a multiple or a part of the ſecond, the third is the ſame multiple or the ſame part of the fourth.

Let $A : B :: C : D$, and firſt let A be a multiple of B, C is the ſame multiple of D; that is, if $A = mB$, $C = mD$.

Take of A and C equimultiples by any number as 2, viz. $2A$ and $2C$; and of B and D, take equimultiples by the number $2m$, viz. $2mB$, $2mD$ [a]; then, becauſe $A = mB$, $2A = 2mB$; and ſince $A : B :: C : D$, and ſince $2A = 2mB$, therefore $2C = 2mD$ [b], and $C = mD$, that is, C contains D m times, or as often as A contains B.

[a] 3. 5.
[b] def. 5. 5.

Next, Let A be a part of B, C is the ſame part of D. For, ſince $A : B :: C : D$, inverſely [c], $B : A :: D : C$. But A being a part of B, B is a multiple of A, and therefore, as is ſhewn above, D is the ſame multiple of C, and therefore C is the ſame part of D that A is of B. Therefore, &c. Q. E. D.

[c] A. 5.

PROP. VII. THEOR.

EQUAL magnitudes have the ſame ratio to the ſame magnitude; and the ſame has the ſame ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other; $A : C :: B : C$.

Let mA, mB be any equimultiples of A and B; and nC any multiple of C.

Becauſe $A = B$, $mA = mB$ [a]; wherefore, if mA be greater than nC, mB is greater than nC [b]; and if $mA = nC$, $mB = nC$; or, if $mA < nC$, $mB < nC$. But mA and mB are any equimultiples of A and B, and nC is any multiple of C, therefore [b] $A : C :: B : C$.

[a] A. 1. 5.
[b] def. 5. 5.

Again, if $A = B$, $C : A :: C : B$; for, as has been proved, $A : C :: B : C$, and inverſely [c], $C : A :: C : B$. Therefore, &c. Q. E. D.

[c] A. 5.

PROP. VIII. PROB.

OF unequal magnitudes, the greater has a greater ratio to the ſame than the leſs has; and the ſame magnitude has a greater ratio to the leſs than it has to the greater.

Let $A+B$ be a magnitude greater han A and C a third magnitude, $A+B$ has to C a greater ratio than A has to C; and C has a greater ratio to A than it has to $A+B$.

Let m be ſuch a number that mA and mB are each of them greater than C; and let nC be the leaſt multiple of C that exceeds $mA+mB$; then $nC-C$, that is, $\overline{n-1}.C$ [a] will be leſs than $mA+mB$, or $mA+mB$, that is $m.\overline{A+B}$ is greater than $\overline{n-1}.C$. But becauſe nC is greater than $mA+mB$, and C leſs than mB, $nC-C$ is greater than mA, or mA is leſs than $nC-C$, that is, than $\overline{n-1}.C$. Therefore the multiple of $A+B$ by m exceeds the multiple of C by $n-1$, but the multiple of A by m does not exceed the multiple of C by $n-1$, therefore $A+B$ has a greater ratio to C than A has to C [b]. — a 1. 5. b 7. def. 5.

Again, becauſe the multiple of C by $n-1$, exceeds the multiple of A by m, but does not exceed the multiple of $A+B$ by m, C has a greater ratio to A than it has to $A+B$ [b]. Therefore, &c. Q. E. D.

PROP. IX. THEOR.

MAGNITUDES which have the ſame ratio to the ſame magnitude are equal to one another; and thoſe to which the ſame magnitude has the ſame ratio are equal to one another.

If $A:C::B:C$, $A=B$.

For if not, let A be greater than B; then, becauſe A is greater than B, two numbers, m and n, may be found as in the laſt propoſition, ſuch that mA ſhall exceed nC, while mB does not exceed nC. But becauſe $A:C::B:C$; if mA exceed nC, mB muſt alſo exceed nC [a]; and it is alſo ſhewn that mB does not exceed nC, which is impoſſible. Therefore A is not greater than B; and in the ſame way it is demonſtrated that B is not greater than A; therefore A is equal to B. — a 5. def. 5.

Next, let $C:A::C:B$, $A=B$. For by inverſion [b], $A:C::B:C$; and therefore by the firſt caſe, $A=B$. — b A. 5.

PROP.

PROP. X. THEOR.

THAT magnitude which has a greater ratio than another has unto the ſame magnitude is the greater of the two: And that magnitude to which the ſame has a greater ratio than it has unto another magnitude is the leſſer of the two.

If the ratio of A to C be greater than that of B to C, A is greater than B.

Becauſe $A : C > B : C$, two numbers m and n may be found, ſuch that $mA > nC$, and $mB < nC$[a]. Therefore alſo $mA > mB$, and $A > B$[b]. a def. 7. 5. b 4. Ax. 5.

Again, let $C : B > C : A$; $B < A$. For two numbers, m and n may be found, ſuch that $mC > mB$, and $mC < mA$[a]. Therefore, ſince mB is leſs, and mA greater than the ſame magnitude, mC, $mB < mA$, and therefore $B < A$.

PROP. XI. THEOR.

RATIOS that are equal to the ſame ratio are equal to one another.

If $A : B :: C : D$; and alſo $C : D :: E : F$; then $A : B :: E : F$.

Take mA, mC, mE any equimultiples of A, C, and E; and nB, nD, nF any equimultiples of B, D, and F. Becauſe $A : B :: C : D$, if $mA > nB$, $mC > nD$[a]; but if $mC > nD$, $mE > nF$[a], becauſe $C : D :: E : F$; therefore if $mA > nB$, $mE > nF$. In the ſame manner, if $mA = nB$, $mE = nF$; and if $mA < nB$, $mE < nF$. Now, mA, mE are any equimultiples whatever of A and E; and nB, nF any whatever of B and F; therefore $A : B :: E : F$[a]. Therefore, &c. Q. E. D. a def. 5. 5.

PROP. XII. THEOR.

IF any number of magnitudes be proportionals, as one of the antecedents is to its conſequent, ſo ſhall all the antecedents taken together be to all the conſequents.

If $A : B :: C : D$, and $C : D :: E : F$; $A : B :: A+C+E : B+D+F$.

Take mA, mC, mE any equimultiples of A, C, and E; and nB, nD, nF, any equimultiples of B, D, and F. Then, be-
a 5. def. 5. cauſe $A : B :: C : D$, if $mA > nB$, $mC > nD$ [a]; and when $mC > nD$, $mE > nF$, becauſe $C : D :: E : F$. Therefore, if $mA > nB$, $mA+mC+mE > nB+nD+nF$: In the ſame manner, if $mA = nB$, $mA+mC+mE = nB+nD+nF$; and if $mA < nB$, $mA+mC+mE < nB+nD+nF$. Now $mA+mC+mE =$
b 1. 5. $m.\overline{A+C+E}$ [b], ſo that mA and $mA+mC+mE$ are any equimultiples of A and of $A+C+E$. And for the ſame reaſon nB, and $nB+nD+nF$ are any equimultiples of B, and of $B+D+F$; therefore [a] $A : B :: A+C+E : B+D+F$. Therefore, &c. Q. E. D.

PROP. XIII. PROB.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth has to the ſixth.

If $A : B :: C : D$; but $C : D > E : F$; then alſo, $A : B > E : F$.

Becauſe $C : D > E : F$, there are ſome two numbers m and n,
a 7. def. 5. ſuch that $mC > nD$, but $mE < nF$ [a]. Now, if $mC > nD$, $mA > nB$, becauſe $A : B :: C : D$. Therefore $mA > nB$, and $mE < nF$, wherefore, $A : B > E : F$. Therefore, &c. Q. E. D.

PROP. XIV. THEOR.

If the firſt has to the ſecond the ſame ratio which the third has to the fourth, then, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; and if leſs, leſs.

If $A : B :: C : D$; then if $A > C$, $B > D$, if $A = C$, $B = D$, and if $A < C$, $B < D$.

Firſt, let $A > C$; then $A : B > C : B$[a], but $A : B :: C : D$, a 8. 5.
therefore $C : D > C : B$[b], and therefore $B > D$[c]. b 13 5.
c 10. 5.

In the ſame manner, it is proved, that if $A = C$, $B = D$; and if $A < C$, $B < D$. Therefore, &c. Q. E. D.

PROP. XV. THEOR.

Magnitudes have the ſame ratio to one another which their equimultiples have.

If A and B be two magnitudes, and m any number, $A : B :: mA : mB$.

Becauſe $A : B :: A : B$[a]; $A : B :: A + A : B + B$[b], or $A :$ a 7. 5.
$B :: 2A : 2B$. And ſince it has been proved that $A : B ::$
$2A : 2B$, $A : B :: A + 2A :: B + 2B$[b], or $A : B :: 3A : 3B$; b 12.
and ſo on, for all the equimultiples of A and B. Therefore, &c. Q. E. D.

PROP. XVI. THEOR.

If four magnitudes of the ſame kind be proportionals, they ſhall alſo be proportionals when taken alternately.

If $A : B :: C : D$, then alternately, $A : C :: B : D$.
Take mA, mB any equimultiples of A and B, and nC, nD
a 15. 5. any equimultiples of C and D. Then[a] $A : B :: mA : mB$;
b 11. 5. now $A : B :: C : D$, therefore[b] $C : D :: mA : mB$. But $C :$
$D :: nC : nD$[a]; therefore $mA : mB :: nC : nD$[b]; wherefo
c 14. 5. if $mA > nC$, $mB > nD$[c]; if $mA = nC$, $mB = nD$, or if mA
nC, $mB < nD$; therefore, $A : C :: B : D$. Therefore, &
Q. E. D.

PROP. XVII. THEOR.

IF magnitudes, taken jointly, be proportionals, they ſhall alſo be proportionals when taken ſeparately; that is, if the firſt, together with the ſecond, have to the ſecond the ſame ratio which the third, together with the fourth, has to the fourth, the firſt ſhall have to the ſecond the ſame ratio which the third has to the fourth.

If $A+B : B :: C+D : D$, then by diviſion $A : B :: C : D$.
Take mA and nB any equimultiples of A and B, by the
numbers m and n; and firſt, let $mA > nB$: to each of them
a 1. 5. add mB, then $mA+mB > mB+nB$. But $mA+mB = m.\overline{A+B}$[a],
b 2. 5. and $mB+nB = \overline{m+n}.B$[b], therefore $m.\overline{A+B} > \overline{m+n}.B$.
And becauſe $A+B : B :: C+D : D$, if $m.\overline{A+B} > \overline{m+n}.B$,
$m.\overline{C+D} > \overline{m+n}.D$, or $mC+mD > mD+nD$, that is, taking
mD from both, $mC > nD$. Therefore, when mA is greater
than nB, mC is greater than nD. In like manner it is de-
monſtrated, that if $mA = nB$, $mC = nD$, and if $mA < nB$, mC
c 5. def. 5. $< nD$; therefore $A : B :: C : D$[c]. Therefore, &c. Q. E. D.

PROP.

PROP. XVIII. THEOR.

IF magnitudes, taken ſeparately, be proportionals, they ſhall alſo be proportionals when taken ointly, that is, if the firſt be to the ſecond as the hird to the fourth, the firſt and ſecond together hall be to the ſecond, as the third and fourth together to the fourth.

If $A:B::C:D$, then, by compoſition, $A+B:B::$ $+D:D$.

Take $m.\overline{A+B}$, and nB any multiples whatever of $A+B$ d B: and firſt, let m be greater than n. Then, becauſe $A+B$ is alſo greater than B, $m.\overline{A+B} > n.B$. For the ſame aſon, $m.\overline{C+D} > nD$. In this caſe, therefore, that is, when $> n$, $m.\overline{A+B}$ is greater than nB, and $m.\overline{C+D}$ is greater an nD. And in the ſame manner it may be proved, that hen $m=n$, $m.\overline{A+B}$ is greater than nB, and $m.\overline{C+D}$ greater han nD.

Next, let $m < n$, or $n > m$, then $m.\overline{A+B}$ may be greater han nB, or may be equal to it, or may be leſs; firſt, let $\overline{A+B}$ be greater than nB; then alſo, $mA+mB > nB$; take B, which is leſs than nB, from both, and $mA > \overline{nB-mB}$, $mA > \overline{n-m}.B$[a]. But if $mA > \overline{n-m}.B$, $mC > \overline{n-m}.D$; a 6. 5.
ecauſe $A:B::C:D$. Now, $\overline{n-m}.D = nD - mD$ [a], erefore, $mC > nD-mD$, and adding mD to both $C+mD > nD$, that is[b] $m.\overline{C+D} > nD$. If therefore, $m.\overline{A+B}$ b 1. 5.
nB, $m.\overline{C+D} > nD$.

In the ſame manner it will be proved, that if $m.\overline{A+B} = nB$, $.\overline{C+D} = nD$; and if $m.\overline{A+B} < nB$, $m.\overline{C+D} < nD$; erefore, $A+B:B::C+D:D$. Therefore, &c. Q. E. D.

PROP. XIX. THEOR.

IF a whole magnitude be to a whole, as a ma tude taken from the firſt, is to a magnitude ken from the other; the remainder ſhall be to remainder, as the whole to the whole.

If A : B :: C : D, and if C be leſs than A, A—C : B :: A : B.

a 16. 5. Becauſe A : B :: C : D, alternately[a], A : C :: B : D;
b 17. 5. therefore by diviſion[b] A—C : C :: B—D : D. Where
again, alternately, A—C : B—D :: C : D; but A : B :: C
therefore A—C : B—D :: A : B. Therefore, &c. Q. E

Cor. A—C : B—D :: C : D.

PROP. D. THEOR.

IF four magnitudes be proportionals, they are proportionals by converſion, that is, the firſt i its exceſs above the ſecond, as the third to its ex above the fourth.

If A : B :: C : D, by converſion, A : A—B :: C : C-
a 17. 5. For ſince A : B :: C : D, by diviſion[a], A—B :
b A. 5. C—D : D, and inverſely[b], B : A—B :: D : C—D; there
c 18. 5. by compoſition[c], A : A—B :: C : C—D. Therefore,
Q. E. D.

PRO

PROP. XX. THEOR.

IF there be three magnitudes, and other three, which, taken two and two, have the ſame ratio; if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

If there be three magnitudes, A, B, and C, and other three D, E, and F; and if $A : B :: D : E$; and alſo $B : C :: E : F$, then if $A > C$, $D > F$; if $A = C$, $D = F$; and if $A < C$, $D < F$.

A, B, C,
D, E, F.

Firſt, let $A > C$; then $A : B > C : B$[a]. But $A : B :: D : E$, a 8. 5.
therefore alſo $D : E > C : B$[b]. Now $B : C :: E : F$, and in- b 13. 5.
verſely[c], $C : B :: F : E$; and it has been ſhewn that $D : E$ c A. 5.
$> C : B$, therefore $D : E > F : E$[b], and conſequently $D > F$[d]. d 10. 5.
Next, let $A = C$; then $A : B :: C : B$[e], but $A : B :: D : E$, e 7. 5.
therefore, $C : B :: D : E$; but $C : B :: F : E$, therefore,
$D : E :: F : E$[f], and $D = F$[g]. Laſtly, let $A < C$. Then f 11. 5.
$C > A$, and, as was ſhewn in the firſt caſe, $C : B :: F : E$, and g 9. 5.
$B : A :: E : D$, therefore, by the firſt caſe, if $C > A$, $F > D$,
that is, if $A < C$, $D < F$. Therefore, &c. Q. E. D.

PROP. XXI. THEOR.

IF there be three magnitudes, and other three, which have the ſame ratio taken two and two, but in a croſs order; if the firſt magnitude be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

If there be three magnitudes, A, B, C, and other three, D, E, and F, ſuch that $A : B :: E : F$, and $B : C :: D : E$; if $A > C$, $D > F$; if $A = C$, $D = F$, and if $A < C$, $D < F$.

Again, let there be four magnitudes, and other four, which, two and two, have the ſame ratio, viz. $A : B :: E : F$; $B : C :: F : G$; $C : D :: G : H$, then $A : D :: E : H$.

For ſince A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the ſame ratio, by the foregoing caſe $A : C :: E : G$. And becauſe alſo $C : D :: G : H$, by that ſame caſe, $A : D :: E : H$. In the ſame manner is the demonſtration extended to any number of magnitudes. Therefore, &c. Q. E. D.

A, B, C, D,
E, F, G, H.

PROP. XXIII. THEOR.

IF there be any number of magnitudes, and as many others, which, taken two and two, in a croſs order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt*.

Firſt, let there be three magnitudes, A, B, C, and other three, D, E, and F, which, taken two and two in a croſs order, have the ſame ratio, viz. $A : B :: E : F$, and $B : C :: D : E$, then $A : C :: D : F$.

Take of A, B, and D any equimultiples mA, mB, mD; and of C, E, F any equimultiples nC, nE, nF.

Becauſe $A : B :: E : F$, and becauſe alſo $A : B :: mA : mB$ [a], and $E : F :: nE : nF$ therefore, $mA : mB :: nE : nF$ [b]. Again, becauſe $B : C :: D : E$, $mB : nC :: mD : nE$; and it has been juſt ſhewn that $mA : mB :: nE : nF$; therefore, if $mA > nC$, $mD > nF$ [c]; if $mA = nC$, $mD = nF$; and if $mA < nC$, $mD < nF$. Now, mA and mD are any equimultiples of A and D, and nC, nF any equimultiples of C and F; therefore, $A : C :: D : F$ [d].

A, B, C,
D, E, F,
mA, mB, nC,
mD, nE, nF.

a 15. 5.
b 11. 5.
c 21. 5.
d 5. def 5.

---

* N. B. This propoſition is uſually cited by the words, "ex æquali in proportione perturbata;" or, "ex æquo perturbatè."

Next, let there be four magnitudes A, B, C, and D, and other four. E, F, G, and H, which, taken two and two in a croſs order, have the ſame ratio, viz. A : B : : G : H; B : C : : F : G, and C : D : : E : F, then A : D : : E : F. For ſince A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, in a croſs order, have the ſame ratio, by the firſt caſe A : C : : E : G. But C : D : : G : H, therefore again, by the firſt caſe, A : D : : E : H. In the ſame manner may the demonſtration be extended to any number of magnitudes. Therefore, &c. Q. E. D.

A,	B,	C,	D.
E,	F,	G,	H.

PROP. XXIV. THEOR.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth; and the fifth to the ſecond, the ſame ratio which the ſixth has to the fourth; the firſt and fifth together ſhall have to the ſecond, the ſame ratio which the third and ſixth together have to the fourth.

Let A : B : : C : D, and alſo E : B : : F : D, then A+E : B : : C+F : D.

Becauſe B : E : : D : F, by inverſion, E : B : : F : D.
a 22. 5. But by hypotheſis, A : B : : C : D, therefore, ex æquali[a],
b 18. 5. A : E : : C : F, and by compoſition[b], A+E : E : : C+F : F.
Now again by hypotheſis, B : E : : D : F, therefore, ex æquali[a],
A+E : B : : C+F : D. Therefore, &c. Q. E. D.

PROP. E. THEOR.

RATIOS which are compounded of the ſame ratios, are the ſame with one another.

Let the ratios of A to B, and of B to C, which compound the ratio of A to C, be equal, each to each, to the ratios of D to E, and E to F, which compound the ratio of E to F, A : C : : E : F.

A, B, C,
D, E, F.

For, firſt, if the ratio of A to B be equal to that of D to E, and the ratio of B to C equal to that of E to F, ex æquali[a], A : C : : D : F. a 22. 5.

And next, if the ratio of A to B be equal to that of E to F, and the ratio of B to C equal to that of D to E, ex æquali in proportione perturbata[b], A : C : : D : F. In the ſame manner may the propoſition be demonſtrated. whatever be the number of ratios. Therefore, &c. Q. E. D. b 23. 5.

ELE-

ELEMENTS

OF

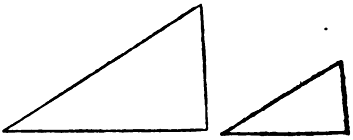
GEOMETRY.

BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are thoſe which have their ſeveral angles equal, each to each, and the ſides about the equal angles proportionals.



II.

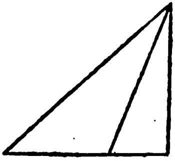
Two ſides of one figure are ſaid to be reciprocally proportional to two ſides of another, when one of the ſides of the firſt is to one of the ſides of the other, as the remaining ſide of the other is to the remaining ſide of the firſt.

III.

A ſtraight line is ſaid to be cut in extreme and mean ratio, when the whole is to the greater ſegment, as the greater ſegment is to the leſs.

IV.

The altitude of any figure is the ſtraight line drawn from its vertex perpendicular to the baſe.



PROP.

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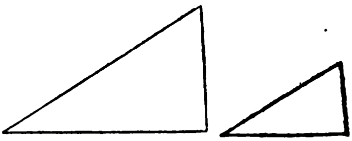

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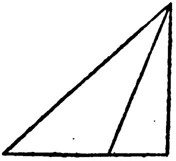
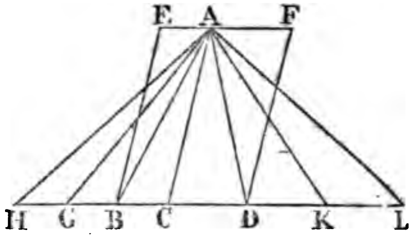


PROP. I. THEOR.

TRIANGLES and parallelograms of the ſame altitude are one to another as their baſes.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the ſame altitude, viz. the perpendicular drawn from the point A to BD: Then, as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways to the points H, L, and take any number of ſtraight lines BG, GH, each equal to the baſe BC; and DK, KL, any number of them, each equal to the baſe CD; and join AG, AH, AK, AL: Then, becauſe CB, BG, GH are all equal, the triangles AHG, AGB, ABC are all
[a] 38. 1. equal [a]: Therefore, whatever multiple the baſe HC is of the baſe BC, the ſame multiple is the triangle AHC of the triangle ABC. For the ſame reaſon, whatever multiple the baſe LC is of the baſe CD, the ſame multiple is the triangle ALC of the triangle ADC: And if the baſe HC be equal to the baſe CL, the triangle AHC is alſo equal to the triangle ALC [a]; and if the baſe HC be greater than the baſe CL, likewiſe the triangle AHC is greater than the triangle ALC; and if leſs, leſs. Therefore, ſince there are four magnitudes, viz. the two baſes BC, CD, and the two triangles ABC, ACD; and of the baſe BC and the triangle ABC, the firſt and third, any equimultiples whatever have been taken, viz. the baſe HC and triangle AHC; and of the baſe CD and triangle ACD, the ſecond and fourth, have been taken any equimultiples whatever, viz. the baſe CL and triangle ALC; and ſince it has been ſhown, that if the baſe HC be greater than the baſe CL, the triangle AHC is greater than the triangle



angle ALC; and if equal, equal; and if leſs, leſs: There-
fore [b], as the baſe BC is to the baſe CD, ſo is the triangle b 5. def. 5.
ABC to the triangle ACD.

And becauſe the parallelogram CE is double of the triangle
ABC [c], and the parallelogram CF double of the triangle c 41. 1.
ACD, and that magnitudes have the ſame ratio which their
equimultiples have [d]; as the triangle ABC is to the triangle d 15. 5.
ACD, ſo is the parallelogram EC to the parallelogram CF.
And becauſe it has been ſhown, that, as the baſe BC is to the
baſe CD, ſo is the triangle ABC to the triangle ACD; and
as the triangle ABC to the triangle ACD, ſo is the parallelo-
gram EC to the parallelogram CF; therefore, as the baſe BC
is to the baſe CD, ſo is [e] the parallelogram EC to the paral- e 11. 5.
lelogram CF. Wherefore triangles &c. Q. E. D.

Cor. From this it is plain, that triangles and parallelograms that have equal altitudes, are one to another as their baſes.

Let the figures be placed ſo as to have their baſes in the
ſame ſtraight line; and having drawn perpendiculars from the
vertices of the triangles to the baſes, the ſtraight line which
joins the vertices is parallel to that in which their baſes are [f], f 33. 1.
becauſe the perpendiculars are both equal and parallel to one
another. Then, if the ſame conſtruction be made as in the
propoſition, the demonſtration will be the ſame.

PROP. II. THEOR.

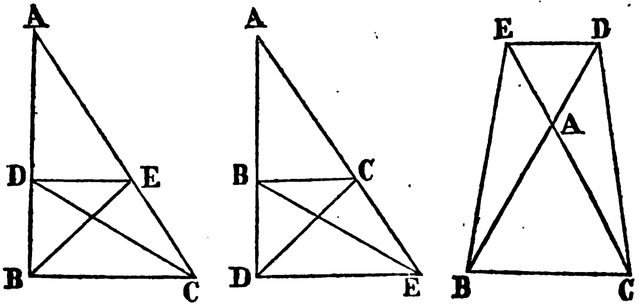
If a ſtraight line be drawn parallel to one of the ſides of a triangle, it ſhall cut the other ſides, or the other ſides produced, proportionally: And if the ſides, or the ſides produced, be cut proportionally, the ſtraight line which joins the points of ſection ſhall be parallel to the remaining ſide of the triangle.

Let DE be drawn parallel to BC, one of the ſides of the triangle ABC: BD is to DA, as CE to EA.

Join BE, CD; then the triangle BDE is equal to the triangle
CDE [a], becauſe they are on the ſame baſe DE, and between a 37. 1.
the ſame parallels DE, BC: but ADE is another triangle,
and

and equal magnitudes have to the ſame the ſame ratio b;
b 7. 5. therefore, as the triangle BDE to the triangle ADE, ſo is the triangle CDE to the triangle ADE; but as the triangle BDE
c 1. 6. to the triangle ADE, ſo is c BD to DA, becauſe having the ſame altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their baſes; and for the ſame reaſon, as the triangle CDE to the triangle ADE, ſo is
d 11. 5. CE to EA. Therefore, as BD to DA, ſo is CE to EA d.

Next, let the ſides AB, AC of the triangle ABC, or theſe



ſides produced, be cut proportionally in the points D, E, that is, ſo that BD be to DA, as CE to EA, and join DE; DE is parallel to BC.

The ſame conſtruction being made, becauſe as BD to DA, ſo is CE to EA; and as BD to DA, ſo is the triangle BDE
e 1. 6. to the triangle ADE e; and as CE to EA, ſo is the triangle CDE to the triangle ADE; therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE have the ſame ratio to the
f 9. 5. triangle ADE; and therefore f the triangle BDE is equal to the triangle CDE: And they are on the ſame baſe DE; but equal triangles on the ſame baſe are between the ſame paral-
g 39. 1. lels g; therefore DE is parallel to BC. Whereſore, if a ſtraight line, &c. Q. E. D.

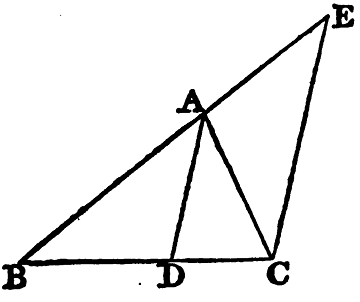
PROP.

PROP. III. THEOR.

IF the angle of a triangle be biſected by a ſtraight line which alſo cuts the baſe; the ſegments of the baſe ſhall have the ſame ratio which the other ſides of the triangle have to one another: And if the ſegments of the baſe have the ſame ratio which the other ſides of the triangle have to one another, the ſtraight line drawn from the vertex to the point of ſection, biſects the vertical angle.

Let the angle BAC of any triangle ABC be divided into two equal angles by the ſtraight line AD: BD is to DC, as BA to AC.

Through the point C draw CE parallel[a] to DA, and let BA produced meet CE in E. Becauſe the ſtraight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD[b]: But CAD, by the hypotheſis, is equal to the angle BAD; wherefore BAD is equal to the angle ACE. Again, becauſe the ſtraight line BAE meets the parallels AD, EC, the outward angle BAD is equal to the inward and oppoſite angle AEC: But the angle ACE has been proved equal to the angle BAD; therefore alſo ACE is equal to the angle AEC, and conſequently the ſide AE is equal to the ſide[c] AC: And becauſe AD is drawn parallel to one of the ſides of the triangle BCE, viz. to EC, BD is to DC, as BA to AE[d]; but AE is equal to AC; therefore, as BD to DC, ſo is BA to AC[e].

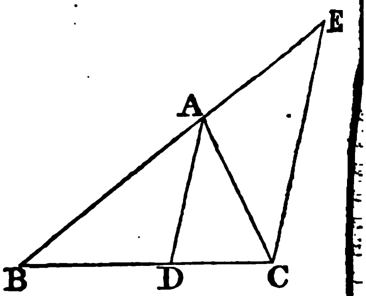


[a] 31. 1.
[b] 29. 1.
[c] 6. 1.
[d] 2. 6.
[e] 7. 5.

Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the ſtraight line AD.

The ſame conſtruction being made; becauſe, as BD to DC, ſo is BA to AC; and as BD to DC, ſo is BA to AE [d], becauſe AD is parallel to EC; therefore AB is to AC, as BA to AE [f]: Conſequently AC is equal to AE [g], and the angle AEC is therefore equal to the angle ACE [h]: But the angle AEC is equal to the outward and oppoſite angle BAD; and the angle ACE is equal to the alternate angle CAD [b]: Wherefore alſo the angle BAD is equal to the angle CAD: Therefore the angle BAC is cut into two equal angles by the ſtraight line AD. Therefore, if the angle, &c. Q. E. D.

f 11. 5.
g 9. 5.
h 5. 1.

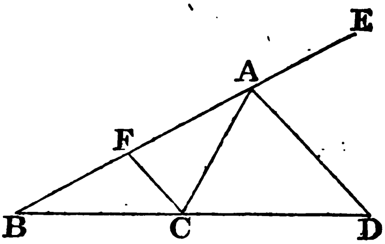


PROP. A. THEOR.

If the exterior angle of a triangle be biſected by a ſtraight line which alſo cuts the baſe produced; the ſegments between the biſecting line and the extremities of the baſe have the ſame ratio which the other ſides of the triangle have to one another; And if the ſegments of the baſe produced have the ſame ratio which the other ſides of the triangle have, the ſtraight line drawn from the vertex to the point of ſection biſects the exterior angle of the triangle.

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the ſtraight line AD which meets the baſe produced in D: BD is to DC, as BA to AC.

Through C draw CF parallel to AD[a]; and becauſe the ſtraight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD[b]: But CAD is equal to the angle DAE[c]; therefore alſo DAE is equal to the angle ACF. Again, becauſe the ſtraight line FAE meets the parallels AD, FC, the outward angle DAE is equal to the inward and oppoſite angle CFA: But the angle ACF has been proved to be equal to the angle DAE; therefore alſo the angle ACF is equal to the angle CFA, and conſequently the ſide AF is equal to the ſide AC[d]: and becauſe AD is parallel to FC, a ſide of the triangle BCF, BD is to DC, as BA to AF[e]; but AF is equal to AC; as therefore BD is to DC, ſo is BA to AC.



[a] 31. 1.
[b] 29. 1.
[c] Hyp.
[d] 6. 1.
[e] 2. 6.

Let now BD be to DC, as BA to AC, and join AD; the angle CAD is equal to the angle DAE.

The ſame conſtruction being made, becauſe BD is to DC, as BA to AC; and alſo BD to DC, as BA to AF[e]; therefore BA is to AC, as BA to AF[f]; wherefore AC is equal to AF[g], and the angle AFC equal[h] to the angle ACF: But the angle AFC is equal to the outward angle EAD, and the angle ACF to the alternate angle CAD; therefore alſo EAD is equal to the angle CAD. Wherefore, if the exterior, &c. Q. E. D.

[f] 11. 5.
[g] 9. 5.
[h] 5. 1.

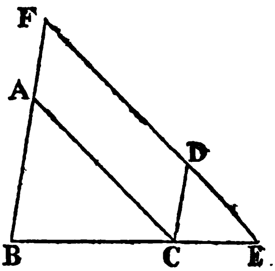
PROP. IV. THEOR.

THE ſides about the equal angles of equiangular triangles are proportionals; and thoſe which are oppoſite to the equal angles are homologous ſides, that is, are the antecedents or conſequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and conſequently[a] the angle BAC equal to the angle CDE. The ſides about the equal angles of the triangles ABC, DCE are proportionals; and thoſe are the homologous ſides which are oppoſite to the equal angles.

[a] 32. 1.

Let the triangle DCE be placed, ſo that its ſide CE may be contiguous to BC, and in the ſame ſtraight line with it: And becauſe the angles ABC, ACB are together leſs than b 17. 1. two right angles [b], ABC and DEC, which is equal to ACB, are alſo leſs than two right angles; wherefore BA, ED c cor. 29. 1. produced ſhall meet [c]; let them be produced and meet in the point F; and becauſe the angle ABC is equal to the angle d 28. 1. DCE, BF is parallel [d] to CD. Again, becauſe the angle ACB is equal to the angle DEC, AC is parallel to FE [d]: Therefore FACD is a parallelogram; and conſequently AF is equal to e 34. 1. CD, and AC to FD [e]: And becauſe AC is parallel to FE, f 2. 6. one of the ſides of the triangle FBE, BA : AF :: BC : CE [f]: g 7. 5. but AF is equal to CD; therefore [g], BA : CD :: BC : CE; and alternately, AB : BC :: DC : CE [b]: Again, becauſe CD is parallel to BF, BC : CE :: FD : DE [f]; but FD is equal to AC; therefore, BC : CE :: AC : DE; and alternately, BC : CA :: CE : ED. Therefore, becauſe it has been proved that AB : BC :: DC : CE; and BC : CA :: CE : ED, ex æquali, BA : AC :: CD : DE. Therefore the ſides, &c. Q. E. D.

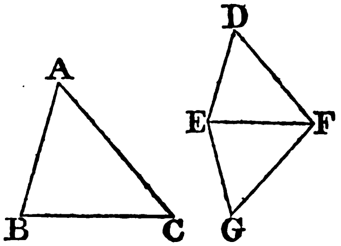


PROP. V. THEOR.

IF the ſides of two triangles, about each of their angles, be proportionals, the triangles ſhall be equiangular, and have their equal angles oppoſite to the homologous ſides.

Let the triangles ABC, DEF have their ſides proportionals, ſo that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and conſequently, ex æquali, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and their equal angles are oppoſite to the homologous ſides, viz. the angle ABC being equal to the angle DEF, and BCA to EFD, and alſo BAC to EDF.

At the points E, F, in the ſtraight line EF, make[a] the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the remaining angle BAC is equal to the remaining angle EGF[b], and the triangle ABC is therefore equiangular to the triangle GEF; and conſequently they have their ſides oppoſite to the equal angles proportionals[c]. Wherefore, as AB to BC, ſo is GE to EF; but as AB to BC, ſo is DE to EF; therefore as DE to EF, ſo[d] GE to EF: Therefore DE and GE have the ſame ratio to EF, and conſequently are equal[e]: For the ſame reaſon, DF is equal to FG: And becauſe, in the triangles DEF, GEF, DE is equal to EG, and EF common, and alſo the baſe DF equal to the baſe GF; therefore the angle DEF is equal[f] to the angle GEF, and the other angles to the other angles, which are ſubtended by the equal ſides[g]. Wherefore the angle DFE is equal to the angle GFE, and EDF to EGF: and becauſe the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF: For the ſame reaſon, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if the ſides, &c. Q. E. D.



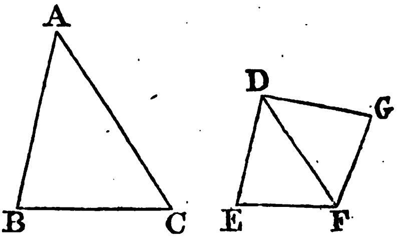
[a] 23. 1.
[b] 32. 1.
[c] 4. 6.
[d] 11. 5.
[e] 9. 5.
[f] 8. 1.
[g] 4. 1.

PROP. VI. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about the equal angles proportionals, the triangles ſhall be equiangular, and ſhall have thoſe angles equal which are oppoſite to the homologous ſides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the ſides about thoſe angles proportionals; that is, BA to AC, as ED to DF; the triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

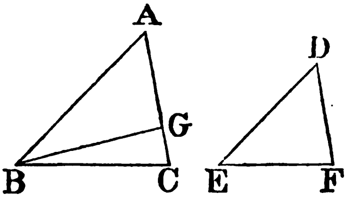
a 23. 1. At the points D, F, in the ſtraight line DF, make[a] the
angle FDG equal to either of the angles BAC, EDF; and
the angle DFG equal to the angle ACB; wherefore the remaining angle at B is equal to the remaining one
b 32. 1. at G[b], and conſequently the triangle ABC is equiangular to the triangle DGF; and therefore as BA to AC, ſo
c 4. 6. is[c] GD to DF: But, by the hypotheſis, as BA to AC, ſo is
d 11. 5 ED to DF; as therefore ED to DF, ſo is[d] GD to DF;
e 9. 5. wherefore ED is equal[e] to DG: And DF is common to the
two triangles EDF, GDF; therefore the two ſides ED,
DF are equal to the two ſides GD, DF; but the angle EDF
is alſo equal to the angle GDF; wherefore the baſe EF is equal
f 4. 1. to the baſe FG[f], and the triangle EDF to the triangle GDF,
and the remaining angles to the remaining angles, each to
each, which are ſubtended by the equal ſides: Therefore the
angle DFG is equal to the angle DFE, and the angle at G to
the angle at E: But the angle DFG is equal to the angle
ACB; therefore the angle ACB is equal to the angle DFE;
g Hyp. and the angle BAC is equal to the angle EDF[g]; wherefore
alſo the remaining angle at B is equal to the remaining angle
at E. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D.



PROP. VII. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about two other angles proportionals, then, if each of the remaining angles be either leſs, or not leſs, than a right angle, the triangles ſhall be equiangular, and have thoſe angles equal about which the ſides are proportionals.

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the ſides about two other angles ABC, DEF proportionals, ſo that AB is to BC, as DE to EF; and in the firſt caſe, let each of the remaining angles at C, F be leſs than a right angle. The triangle ABC is equiangular to the triangle DEF, that is, the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.



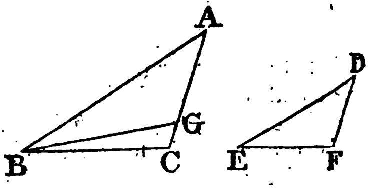
For, if the angles ABC, DEF be not equal, one of them is greater than the other: Let ABC be the greater, and at the point B, in the ſtraight line AB, make the angle ABG equal to the angle [a] DEF: And becauſe the angle at A is equal to the angle at D, and the angle ABG to the angle DEF; the remaining angle AGB is equal [b] to the remaining angle DFE: Therefore the triangle ABG is equiangular to the triangle DEF; wherefore [c], as AB is to BG, ſo is DE to EF; but as DE to EF, ſo, by hypotheſis, is AB to BC: therefore as AB to BC, ſo is AB to BG [d]; and becauſe AB has the ſame ratio to each of the lines BC, BG; BC is equal [e] to BG, and therefore the angle BGC is equal to the angle BCG [f]: But the angle BCG is, by hypotheſis, leſs than a right angle; therefore alſo the angle BGC is leſs than a right angle, and

a 23. 1.
b 32. 1.
c 4. 6.
d 11. 5.
e 9. 5.
f 6. 1.

and the adjacent angle AGB muſt be greater than a right angle[g]. But it was proved that the angle AGB is equal to the angle at F; therefore the angle at F is greater than a right angle: But, by the hypotheſis, it is leſs than a right angle; which is abſurd. Therefore the angles ABC, DEF are not unequal, that is, they are equal: And the angle at A is equal to the angle at D; wherefore the remaining angle at C is equal to the remaining angle at F: Therefore the triangle ABC is equiangular to the triangle DEF.

[g] 13. 1.

Next, let each of the angles at C, F be not leſs than a right angle; the triangle ABC is alſo in this caſe equiangular to the triangle DEF.



The ſame conſtruction being made, it may be proved in like manner that BC is equal to BG, and the angle at C equal to the angle BGC: But the angle at C is not leſs than a right angle; therefore the angle BGC is not leſs than a right angle: Wherefore, two angles of the triangle BGC are together not leſs than two right angles, which is impoſſible[h]; and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the firſt caſe.

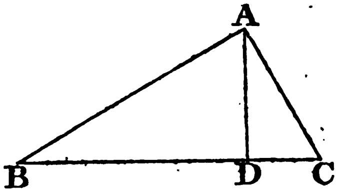
[h] 17. 1.

PROP. VIII. THEOR.

IN a right angled triangle, if a perpendicular be drawn from the right angle to the baſe; the triangles on each ſide of it are ſimilar to the whole triangle, and to one another.

Let ABC be a right angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the baſe BC: the triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another.

Becauſe the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common



mon to the two triangles ABC, ABD; the remaining angle ACB is equal to the remaining angle BAD [a]: therefore the triangle ABC is equiangular to the triangle ABD, and the ſides about their equal angles are proportionals [b]; wherefore the triangles are ſimilar [c]: in the like manner it may be demonſtrated, that the triangle ADC is equiangular and ſimilar to the triangle ABC: And the triangles ABD, ADC, being both equiangular and ſimilar to ABC, are equiangular and ſimilar to each other. Therefore, in a right angled, &c. Q. E. D.

a 32. 1.

b 4. 6.

c 1. def. 6.

Cor. From this it is manifeſt, that the perpendicular drawn from the right angle of a right angled triangle to the baſe, is a mean proportional between the ſegments of the baſe: and alſo that each of the ſides is a mean proportional between the baſe, and its ſegment adjacent to that ſide. For in the triangles BDA, ADC, BD : DA : : DA : DC [b]; and in the triangles ABC, DBA, BC : BA : : BA : BD [b]; and in the triangles ABC, ACD, BC : CA : : CA : CD [b].

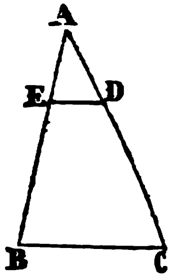
PROP. IX. PROB.

FROM a given ſtraight line to cut off any part required, that is, a part of which the line ſhall be a given multiple.

Let AB be the given ſtraight line; it is required to cut off from it a part of which AB ſhall be a given multiple.

From the point A draw a ſtraight line AC making any angle with AB; and in AC take any point D, and take AC

the ſame multiple of AD, that AB is
of the part which is to be cut off from
it; join BC, and draw DE parallel to it:
then AE is the part required to be cut
off.



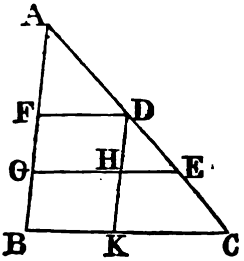
Becauſe ED is parallel to one of the
ſides of the triangle ABC, viz. to BC,
a 2. 6. as CD is to DA, ſo is[a] BE to EA; and,
b 18. 5. by compoſition[b], CA is to AD, as BA
to AE: but CA is a multiple of AD;
c C. 5. therefore[c] BA is the ſame multiple of
AE: whatever part therefore AD is of AC, AE is the
ſame part of AB; wherefore, from the ſtraight line AB the
part required is cut off. Which was to be done.

PROP. X. PROB.

To divide a given ſtraight line ſimilarly to a given divided ſtraight line, that is, into parts that ſhall have the ſame ratios to one another which the parts of the divided given ſtraight line have.

Let AB be the ſtraight line given to be divided, and AC the divided line; it is required to divide AB ſimilarly to AC.

Let AC be divided in the points D, E; and let AB, AC
be placed ſo as to contain any angle, and join BC, and through
a 31. 1. the points D, E, draw[a] DF, EG parallel to BC; and through
D draw DHK parallel to AB; therefore each of the figures
b 34. 1. FH, HB, is a parallelogram; wherefore DH is equal[b] to
FG, and HK to GB: and becauſe
HE is parallel to KC, one of the
ſides of the triangle DKC, as CE
c 2. 6. to ED, ſo is[c] KH to HD: But
KH is equal to BG, and HD to GF;
therefore, as CE to ED, ſo is BG
to GF: again, becauſe FD is parallel to EG, one of the ſides of the
triangle AGE, as ED to DA, ſo is
GF to FA: but it has been proved



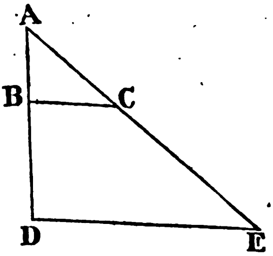
that CE is to ED, as BG to GF; and as ED to DA, ſo GF to FA: therefore the given ſtraight line AB is divided ſimilarly to AC. Which was to be done.

PROP. XI. PROB.

TO find a third proportional to two given ſtraight lines.

Let AB, AC be the two given ſtraight lines, and let them be placed ſo as to contain any angle; it is required to find a third proportional to AB, AC.

Produce AB, AC to the points D, E; and make BD equal to AC; and having joined BC, through D, draw DE parallel to it [a].



Becauſe BC is parallel to DE, a ſide of the triangle ADE, AB is [b] to BD, as AC to CE: but BD is equal to AC; as therefore AB to AC, ſo is AC to CE. Wherefore to the two given ſtraight lines AB, AC a third proportional CE is found. Which was to be done.

a 31. 1.

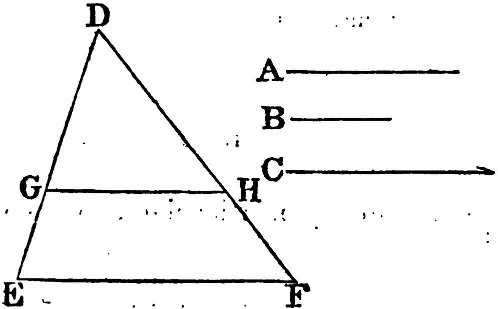
b 2. 6.

PROP. XII. PROB.

TO find a fourth proportional to three given ſtraight lines.

Let A, B, C be the three given ſtraight lines; it is required to find a fourth proportional to A, B, C.

Take two ſtraight lines DE, DF, containing any angle EDF; and upon theſe make DG equal to A, GE equal to B, Book VI. B, and DH equal to C; and having joined GH, draw EF



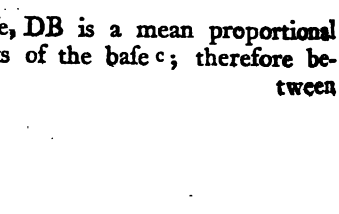
a 31. 1. parallel [a] to it through the point E: and becauſe GH is parallel to EF, one of the ſides of the triangle DEF, DG is to
b 2. 6. GE, as DH to HF [b]; but DG is equal to A, GE to B, and DH to C; therefore, as A is to B, ſo is C to HF. Wherefore to the three given ſtraight lines, A, B, C a fourth proportional HF is found. Which was to be done.

PROP. XIII. PROB.

TO find a mean proportional between two given ſtraight lines.

Let AB, BC be the two given ſtraight lines; it is required to find a mean proportional between them.

Place AB, BC in a ſtraight line, and upon AC deſcribe the ſemicircle ADC, and from
a 11. 1. the point B draw [a] BD at right angles to AC, and join AD, DC.



Becauſe the angle ADC in a ſemicircle is a right
b 31. 3. angle [b], and becauſe in the right angled triangle ADC, DB is drawn from the right angle perpendicular to the baſe, DB is a mean proportional
c Cor. 8. 6. between AB, BC the ſegments of the baſe [c]; therefore between

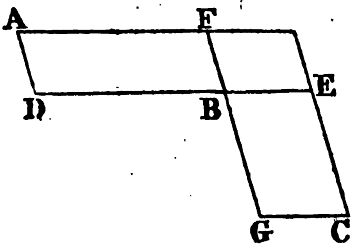
tween the two given ſtraight lines AB, BC, a mean proportional DB is found. Which was to be done.

PROP. XIV. THEOR.

EQUAL parallelograms which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional: And parallelograms that have one angle of the one equal to one angle of the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms, which have the angles at B equal, and let the ſides DB, BE be placed in the ſame ſtraight line; wherefore alſo FB, BG are in one ſtraight line [a]: the ſides of the parallelograms AB, BC about the equal angles, are reciprocally proportional; that is, DB is to BE, as GB to BF. a 14. 1.

Complete the parallelogram FE; and becauſe the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to FE [b]; but as AB to FE, ſo is the baſe DB to BE [c]; and, as BC to FE, ſo is the baſe GB to BF; therefore, as DB to BE, ſo is GB to BF [d]. Wherefore, the ſides of the parallelograms AB, BC about their equal angles are reciprocally proportional. b 7. 5. c 1. 6. d 11. 5.



But, let the ſides about the equal angles be reciprocally proportional, viz. as DB to BE, ſo GB to BF; the parallelogram AB is equal to the parallelogram BC.

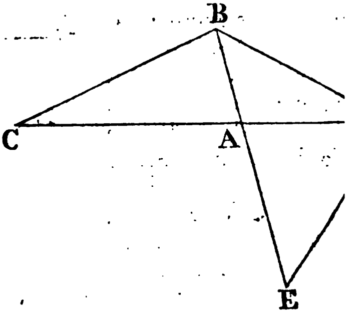
Becauſe, as DB to BE, ſo is GB to BF; and as DB to BE, ſo is the parallelogram AB to the parallelogram FE; and as GB to BF, ſo is the parallelogram BC to the parallelogram FE; therefore as AB to FE, ſo BC to FE [d]: wherefore the parallelogram e 9. 5. parallelogram AB is equal [e] to the parallelogram BC, fore equal parallelograms, &c. Q. E. D.

PROP. XV. THEOR.

EQUAL triangles which have one angle o one equal to one angle of the other, have ſides about the equal angles reciprocally propo al: And triangles which have one angle in the equal to one angle in the other, and their ſid bout the equal angles reciprocally proportiona equal to one another.

Let ABC, ADE be equal triangles, which have the BAC equal to the angle DAE; the ſides about the angles of the triangles are reciprocally proportional; t CA is to AD, as EA to AB.

Let the triangles be placed ſo that their ſides CA, A in one ſtraight line; wherefore alſo EA and AB
a 14. 1. one ſtraight line [a]; join BD. Becauſe the triangle A equal to the triangle ADE, and that ABD is another triangle; therefore as the triangle CAB is to the triangle BAD, ſo is triangle EAD to tri-
b 7. 5. angle DAB [b]: but as triangle CAB to triangle BAD, ſo is the
c 1. 6. baſe CA to AD [c]; and as triangle EAD to triangle DAB, ſo is the baſe EA to AB [c]; as therefore CA to AD, ſo
d 11. 5. to AB [d]; wherefore the ſides of the triangles ABC, about the equal angles are reciprocally proportional.



But let the ſides of the triangles ABC, ADE abc equal angles be reciprocally proportional, viz. CA to

A to AB; the triangle ABC is equal to the triangle DE.

Having joined BD as before; becauſe, as CA to AD, ſo is A to AB; and as CA to AD, ſo is triangle ABC to tri- ngle BAD [c]; and as EA to AB, ſo is triangle EAD to c 1. 6.
iangle BAD [c]; therefore [d] as triangle BAC to triangle d 11. 5.
AD, ſo is triangle EAD to triangle BAD; that is, the ngles BAC, EAD have the ſame ratio to the triangle AD: wherefore the triangle BAC is equal [e] to the triangle e 9. 5.
DE. Therefore equal triangles, &c. Q. E. D.

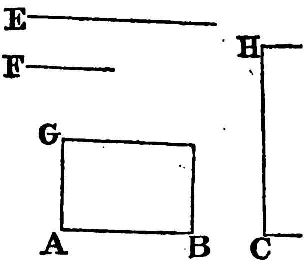
PROP. XVI. THEOR.

IF four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the ctangle contained by the means: And if the rect- gle contained by the extremes be equal to the ctangle contained by the means, the four ſtraight nes are proportionals.

Let the four ſtraight lines, AB, CD, E, F be proportionals, iz. as AB to CD, ſo E to F; the rectangle contained by B, F is equal to the rectangle contained by CD, E.

From the points A, C draw [a] AG, CH at right angles to a 11. 1.
AB, CD; and make AG equal to F, and CH equal to E, nd complete the parallelograms BG, DH: becauſe, as AB o CD, ſo is E to F; and that E is equal to CH, and F to AG; AB is [b] to CD, as CH to AG: therefore the ſides of b 7. 5.
he parallelograms BG, DH about the equal angles are reciprocally proportional; but parallelograms which have their ides about equal angles reciprocally proportional, are equal to ne another [c]; therefore the parallelogram BG is equal to c 14. 6.

the parallelogram DH : and the parallelogram BG contained by the ſtraight lines AB, F ; becauſe AG is equal to F ; and the parallelogram DH is contained by CD and E ; becauſe CH is equal to E : therefore the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD and E.



And if the rectangle contained by the ſtraight lines
F be equal to that which is contained by CD, E ; theſ
lines are proportionals, viz. AB is to CD, as E to F.

The ſame conſtruction being made, becauſe the recta
contained by the ſtraight lines AB, F is equal to that w
is contained by CD, E, and that the rectangle BG is
tained by AB, F, becauſe AG is equal to F ; and the
angle DH by CD, E, becauſe CH is equal to E ; there
the parallelogram BG is equal to the parallelogram DH ;
they are equiangular : but the ſides about the equal angl
d 14. 6. equal parallelograms are reciprocally proportional[d] : w
fore, as AB to CD, ſo is CH to AG ; and CH is equal t
and AG to F : as therefore AB is to CD, ſo E to F. W
fore, if four, &c. Q. E. D.

PROP. XVII. THEOR.

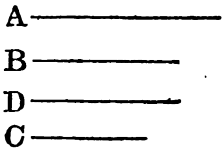
IF three ſtraight lines be proportionals, the r
angle contained by the extremes is equal to
ſquare of the mean : And if the rectangle cont
ed by the extremes be equal to the ſquare of
mean, the three ſtraight lines are proportionals.

Let the three ſtraight lines A, B, C be proportionals,
as A to B, ſo B to C ; the rectangle contained by A, C
qual to the ſquare of B.

D equal to B; and becauſe as A to B, ſo B to C, t B is equal to D; A is [a] to B, as D to C: but if four lines be proportionals, the rectangle contained by the es is equal to that which ained by the means [b]: e the rectangle contained C is equal to that contain- B, D: but the rectangle ed by B, D is the ſquare becauſe B is equal to D: e the rectangle contained C is equal to the ſquare of B.

a 7. 5.

b 16. 6.



if the rectangle contained by A, C be equal to the of B; A is to B, as B to C.

ſame conſtruction being made, becauſe the rectangle ed by A, C is equal to the ſquare of B, and the ſquare equal to the rectangle contained by B, D, becauſe B to D; therefore the rectangle contained by A, C is o that contained by B, D: but if the rectangle con- by the extremes be equal to that contained by the the four ſtraight lines are proportionals [b]: therefore A as D to C; but B is equal to D; wherefore as A to to C. Therefore, if three ſtraight lines, &c. Q. E. D.

PROP. XVIII. PROB.

ON a given ſtraight line to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated to n rectilineal figure.

AB be the given ſtraight line, and CDEF the given eal figure of four ſides; it is required upon the given line AB to deſcribe a rectilineal figure ſimilar, and ſi- ſituated to CDEF.

DF, and at the points A, B in the ſtraight line AB, the angle BAG equal to the angle at C, and the [a] ABG equal to the angle CDF; therefore the remain- le CFD is equal to the remaining angle AGB [b]: ore the triangle FCD is equiangular to the triangle GAB:

a 23. 1.

b 32. 1.

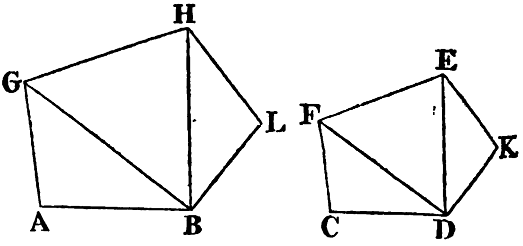
GAB: again, at the points G, B in the ſtraight line GB make [a] the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH: then, becauſe the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE: for the ſame reaſon, the angle ABH is equal to the angle CDE; alſo the angle at A is equal to the angle at C, and the angle GHB to FED: therefore the rectilineal figure ABHG is equiangular to CDEF: but likewiſe theſe figures have their ſides about the equal angles proportionals: becauſe the triangles GAB, FCD being equiangular, BA is [c] to AG, as DC to CF; and becauſe AG is to GB, as CF to FD; and as GB to GH, ſo, by reaſon of the equiangular triangles BGH, DFE, is FD to FE; therefore, ex aequali [d], AG is to GH, as CF to FE: in the ſame manner it may be proved that AB is to BH, as CD to DE: and GH is to HB, as FE to ED [c]. Wherefore, becauſe the rectilineal figures ABHG, CDEF are equiangular, and have their ſides about the equal angles proportionals, they are ſimilar to one another [e].

a 23. 1.

c 4. 6.

d 22. 5.

e 1. def. 6.



Next, Let it be required to deſcribe upon a given ſtraight line AB, a rectilineal figure ſimilar, and ſimilarly ſituated to the rectilineal figure CDKEF.

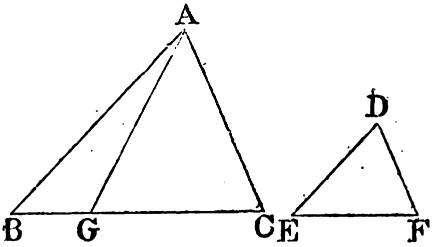
Join DE, and upon the given ſtraight line AB deſcribe the rectilineal figure ABHG ſimilar, and ſimilarly ſituated to the quadrilateral figure CDEF, by the former caſe; and at the points B, H in the ſtraight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L: and becauſe the figures ABHG, CDEF are ſimilar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK: for the ſame reaſon the angle ABL is equal to the angle CDK: therefore the five ſided figures AGHLB, CFEKD are equiangular; and becauſe the figures AGHB, CFED are ſimilar, GH is to HB, as FE to ED; and as HB to HL, ſo is ED to EK[c]; c 4. 6. therefore, ex æquali[d], GH is to HL, as FE to EK: for the d 22. 5. ſame reaſon, AB is to BL, as CD to DK: and BL is to LH, as[c] DK to KE, becauſe the triangles BLH, DKE are equiangular: therefore, becauſe the five ſided figures AGHLB, CFEKD are equiangular, and have their ſides about the equal angles proportionals, they are ſimilar to one another: and in the ſame manner a rectilineal figure of ſix or more ſides may be deſcribed upon a given ſtraight line ſimilar to one given, and ſo on. Which was to be done.

PROP. XIX. THEOR.

SIMILAR triangles are to one another in the duplicate ratio of their homologous ſides.

Let ABC, DEF be ſimilar triangles, having the angle B equal to the angle E, and let AB be to BC, as DE to EF, ſo that the ſide BC is homologous to EF[a]: the triangle ABC a 13. def. 5. has to the triangle DEF, the duplicate ratio of that which BC has to EF.

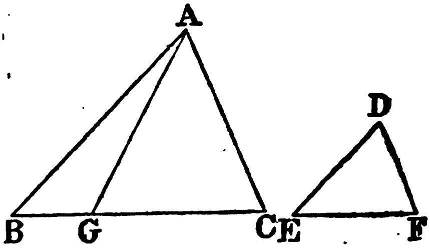
Take BG a third proportional to BC, EF[b], ſo that BC is b 11. 6. to EF, as EF to BG, and join GA: then, becauſe as AB to BC, ſo DE to EF; alternately[c], AB is c 16. 5. to DE, as BC to EF: but as BC to EF, ſo is EF to BG; therefore[d] as d 11. 5. AB to DE, ſo is EF to BG: wherefore the ſides of the triangles ABG, DEF which are about the equal angles, are reciprocally proportional: but triangles which have the ſides a-



bout two equal angles reciprocally proportional are equal to one another e: therefore the triangle ABG is equal to the triangle DEF; and becauſe as BC is to EF, ſo EF to BG; and that if three ſtraight lines be proportionals, the firſt has to the third the duplicate ratio of that which it has to the ſecond; BC therefore has to BG the duplicate ratio of that which BC has to EF: but as BC to BG, ſo is f the triangle ABC to the triangle ABG. Therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF: but the triangle ABG is equal to the triangle DEF; wherefore alſo the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore ſimilar triangles, &c. Q. E. D.

e 15. 6.

f 1. 6.



Cor. From this it is manifeſt, that if three ſtraight lines be proportionals, as the the firſt is to the third, ſo is any triangle upon the firſt to a ſimilar, and ſimilarly deſcribed triangle upon the ſecond.

PROP. XX. THEOR.

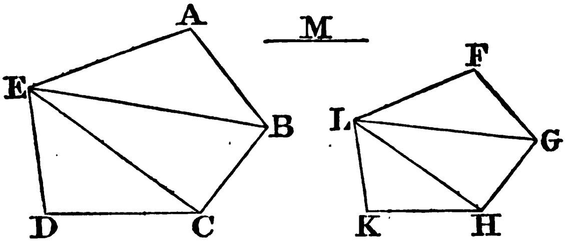
SIMILAR polygons may be divided into the ſame number of ſimilar triangles, having the ſame ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous ſides have.

Let ABCDE, FGHKL be ſimilar polygons, and let AB be the homologous ſide to FG: the polygons ABCDE, FGHKL may be divided into the ſame number of ſimilar triangles, whereof each to each has the ſame ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the ſide FG.

Join BE, EC, GL, LH: and becauſe the polygon ABCDE is ſimilar to the polygon FGHKL, the angle BAE is equal to the angle GFL a, and BA is to AE, as GF to FL a: wherefore, becauſe the triangles ABE, FGL have an angle in one equal to an angle in the other, and their ſides about theſe

a 1. def. 6.

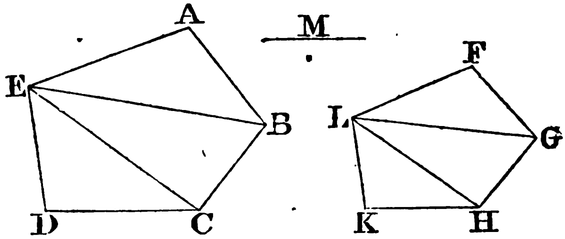
theſe equal angles proportionals, the triangle ABE is equiangular [b],
and therefore ſimilar to the triangle FGL [c]; where- b 6. 6.
fore the angle ABE is equal to the angle FGL: and, be- c 4. 6.
cauſe the polygons are ſimilar, the whole angle ABC is equal
[a] to the whole angle FGH; therefore the remaining angle
EBC is equal to the remaining angle LGH: and becauſe the
triangles ABE, FGL are ſimilar, EB is to BA, as LG to
GF [a]; and alſo, becauſe the polygons are ſimilar, AB is to
BC, as FG to GH [a]; therefore, ex æquali [d], EB is to BC d 22. 5.
as LG to GH; that is, the ſides about the equal angles
EBC, LGH are proportionals; therefore [b] the triangle EBC
is equiangular to the triangle LGH, and ſimilar to it [c]. For
the ſame reaſon, the triangle ECD likewiſe is ſimilar to the
triangle LHK; therefore the ſimilar polygons ABCDE,
FGHKL are divided into the ſame number of ſimilar triangles.



Alſo theſe triangles have, each to each, the ſame ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the conſequents FGL, LGH, LHK: and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the homologous ſide FG.

Becauſe the triangle ABE is ſimilar to the triangle FGL,
ABE has to FGL the duplicate ratio [e] of that which the e 19. 6.
ſide BE has to the ſide GL: for the ſame reaſon, the triangle
BEC has to GLH the duplicate ratio of that which BE has
to GL: therefore, as the triangle ABE to the triangle FGL,
ſo [f] is the triangle BEC to the triangle GLH. Again, be- f 11. 5.
cauſe the triangle EBC is ſimilar to the triangle LGH, EBC
has to LGH the duplicate ratio of that which the ſide EC
has to the ſide LH: for the ſame reaſon, the triangle ECD
has to the triangle LHK, the duplicate ratio of that which
EC has to LH: as therefore the triangle EBC to the triangle
LGH, ſo is [f] the triangle ECD to the triangle LHK: but it
has been proved, that the triangle EBC is likewiſe to the

triangle LGH, as the triangle ABE to the triangle FGL. Therefore, as the triangle ABE is to the triangle FGL, ſo is triangle EBC to triangle LGH, and triangle ECD to triangle LHK: and therefore, as one of the antecedents to one



of the conſequents, ſo are all the antecedents to all the conſequents g. Wherefore, as the triangle ABE to the triangle FGL, ſo is the polygon ABCDE to the polygon FGHKL: but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the ſide AB has to the homologous ſide FG. Therefore alſo the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous ſide FG. Wherefore ſimilar polygons, &c. Q. E. D.

g 12. 5.

Cor. 1. In like manner it may be proved, that ſimilar four ſided figures, or of any number of ſides, are one to another in the duplicate ratio of their homologous ſides, and it has already been proved in triangles. Therefore, univerſally ſimilar rectilineal figures are to one another in the duplicate ratio of their homologous ſides.

Cor. 2. And if to AB, FG, two of the homologous ſides, a third proportional M be taken, AB has h to M the duplicate ratio of that which AB has to FG: but the four ſided figure or polygon upon AB has to the four ſided figure or polygon upon FG likewiſe the duplicate ratio of that which AB has to FG: therefore, as AB is to M, ſo is the figure upon AB to the figure upon FG, which was alſo proved in triangles i. Therefore, univerſally, it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any rectilineal figure upon the firſt, to a ſimilar and ſimilarly deſcribed rectilineal figure upon the ſecond.

h 11. def. 5.
i Cor. 19. 6.

Cor. 3. Becauſe all ſquares are ſimilar figures, the ratio of any two ſquares to one another is the ſame with the duplicate ratio of their ſides; and hence, alſo, any two ſimilar rectilineal figures are to one another as the ſquares of their homologous ſides.

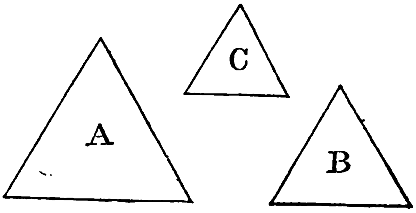
PROP.

PROP. XXI. THEOR.

RECTILINEAL figures which are ſimilar to the ſame rectilineal figure, are alſo ſimilar to one another.

Let each of the rectilineal figures A, B be ſimilar to the rectilineal fignre C: The figure A is ſimilar to the figure B.

Becauſe A is ſimilar to C, they are equiangular, and alſo have their ſides about the equal angles proportionals[a]. Again, a 1. def. 6. becauſe B is ſimilar to C, they are equiangular, and have their ſides about the equal angles proportionals[a]: therefore the figures A, B are each of them equiangular to C, and have the ſides about the equal angles of each of them and of C proportionals.



Wherefore the rectilineal figures A and B are equiangular[b], b 1. Ax. 1. and have their ſides about the equial angles proportionals[c]. c 11. 5. Therefore A is ſimilar[a] to B. Q. E. D.

PROP. XXII. THEOR.

IF four ſtraight lines be proportionals, the ſimilar rectilineal figures ſimilarly deſcribed upon them ſhall alſo be proportionals; and if the ſimilar rectilineal figures ſimilarly deſcribed upon four ſtraight lines be proportionals, thoſe ſtraight lines ſhall be proportionals.

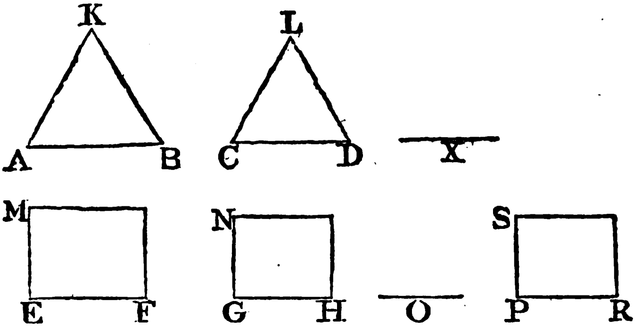
Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH, and upon AB, CD

Book VI. let the ſimilar rectilineal figures KAB, LCD be ſimilarly deſcribed; and upon EF GH the ſimilar rectilineal figures MF, NH in like manner: the rectilineal figure KAB is to LCD, as MF to NH.

a 11. 6. To AB, CD take a third proportional[a] X; and to EF, GH a third proportional O: and becauſe AB is to CD, as
b 11. 5. EF to GH, and that CD is [b] to X, as GH to O; wherefore,
c 22 5. ex æquali [c], as AB to X, ſo EF to O: but as AB to X, ſo
d 2. cor. 20. 6. is [d] the rectilineal KAB to the rectilineal LCD, and as EF to O, ſo is [d] the rectilineal MF to the rectilineal NH: therefore,
d 2. cor. 20. 6. as KAB to LCD, ſo [b] is MF to NH.

And if the rectilineal KAB be to LCD, as MF to NH; the ſtraight line AB is to CD, as EF to GH.

e 12. 6. Make [e] as AB to CD, ſo EF to PR, and upon PR deſcribe
f 18. 6. [f] the rectilineal figure SR ſimilar and ſimilarly ſituated to either of the figures MF, NH: then, becauſe as AB to CD, ſo is EF to PR, and that upon AB, CD are deſcribed the ſimilar and ſimilarly ſituated rectilineals KAB, LCD, and upon EF, PR, in like manner, the ſimilar rectilineals MF, SR; KAB is to LCD, as MF to SR; but, by the hypotheſis, KAB is to LCD, as MF to NH; and therefore the rectilineal MF having the ſame ratio to each of the two NH, SR, theſe
g 9. 5. are equal [g] to one another: they are alſo ſimilar, and ſimilarly ſituated; therefore GH is equal to PR: and becauſe as AB to CD, ſo is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH. If therefore four ſtraight lines, &c. Q. E. D.



PROP.

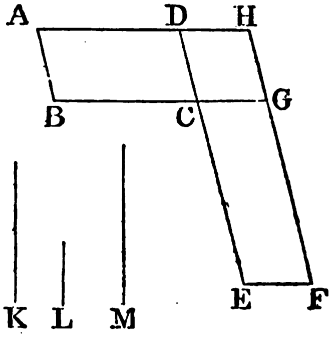
PROP. XXIII. THEOR.

EQUIANGULAR parallelograms have to one another the ratio which is compounded of the ratios of their ſides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG: the ratio of the parallelogram AC to the parallelogram CF, is the ſame with the ratio which is compounded of the ratios of their ſides.

Let BC, CG be placed in a ſtraight line; therefore DC and CE are alſo in a ſtraight line [a]; and complete the parallelogram DG; and, taking any ſtraight line K, make [b] as BC to CG, ſo K to L; and as DC to CE, ſo make [b] L to M: therefore the ratios of K to L, and L to M, are the ſame with the ratios of the ſides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is ſaid to be compounded [c] of the ratios of K to L, and L to M: wherefore alſo K has to M the ratio compounded of the ratios of the ſides: and becauſe as BC to CG, ſo is the parallelogram AC to the parallelogram CH [d]; but as BC to CG, ſo is K to L; therefore K is [e] to L, as the parallelogram AC to the parallelogram CH: again, becauſe as DC to CE, ſo is the parallelogram CH to the parallelogram CF; but as DC to CE, ſo is L to M; wherefore L is [e] to M, as the parallelogram CH to the parallogram CF: therefore, ſince it has been proved, that as K to L, ſo is the parallelogram AC to the parallelogram CH; and as L to M, ſo the parallelogram CH to the parallelogram CF; ex æquali [f], K is to M, as the parallelogram AC to the parallelogram CF:

a 14. 1.
b 12. 6.
c 10. def. 5.
d 1. 6.
e 11. 5.
f 22. 5.



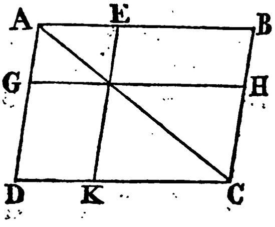
CF: but K has to M the ratio which is compounded of the ratios of the ſides; therefore alſo the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the ſides. Wherefore equiangular parallelograms, &c. Q. E. D.

PROP. XXIV. THEOR.

THE parallelograms about the diameter of any parallelogram, are ſimilar to the whole, and to one another.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter: The parallelograms EG, HK are ſimilar both to the whole parallelogram ABCD, and to one another.

Becauſe DC, GF are parallels, the angle ADC is equal[a] to the angle AGF: for the ſame reaſon, becauſe BC, EF are parallels, the angle ABC is equal to the angle AEF: and each of the angles BCD, EFG is equal to the oppoſite angle DAB[b], and therefore are equal to one another, wherefore the parallelograms ABCD, AEFG are equiangular: and becauſe the angle ABC is equal to the angle AEE, and the angle BAC common to the two triangles BAC, EAF, they are equiangular to one another; therefore[c] as AB to BC, ſo is AE to EF: and becauſe the oppoſite ſides of parallelograms are equal to one another[b], AB is[d] to AD, as AE to AG; and DC to CB, as GF to FE; and alſo CD to DA, as FG to GA: therefore the ſides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefore ſimilar to one another[e]: for the ſame reaſon, the parallelogram ABCD is ſimilar to the parallelogram FHCK. Wherefore each of the parallelograms, GE, KH is ſimilar to DB: but rectilineal figures which are ſimilar to the ſame rectilineal fi-



gure,

[a] 29. 1.

[b] 34. 1.

[c] 4. 6.

[d] 7. 5.

[e] 1. def. 6.

gure, are alſo ſimilar to one another[f]; therefore the parallelogram
GE is ſimilar to KH. Wherefore the parallelograms, f 21. 6.
&c. Q. E. D.

PROP. XXV. PROB.

TO deſcribe a rectilineal figure which ſhall be ſimilar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be deſcribed is required to be ſimilar, and D that to which it muſt be equal. It is required to deſcribe a rectilineal figure ſimilar to ABC, and equal to D.

Upon the ſtraight line BC deſcribe[a] the parallelogram BE a cor. 45. 1.
equal to the figure ABC; alſo upon CE deſcribe[a] the parallelogram
CM equal to D, and having the angle FCE equal
to the angle CBL: therefore BC and CF are in a ſtraight
line[b], as alſo LE and EM between BE and BF find[c] a b {29. 1.
mean proportional GH, and upon GH deſcribe[d] the rectilineal {14. 1.
c 13. 6.
d 18. 6.

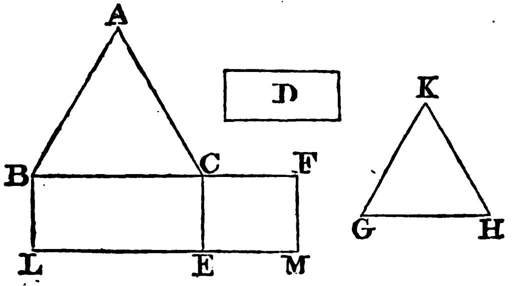


figure KGH ſimilar and ſimilarly ſituated to the figure
ABC: and becauſe BC is to GH as GH to CF, and if three
ſtraight lines be proportional, as the firſt is to the third, ſo is
[e] the figure upon the firſt to the ſimilar and ſimilarly deſcribed e 2. cor.
figure upon the ſecond; therefore as BC to CF, ſo is 20. 6.
the rectilineal figure ABC to KGH: but as BC to CF, ſo is
[f] the parallelogram BE to the parallelogram EF: therefore f 1. 6.

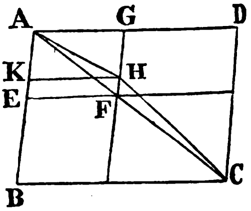
as the rectilineal figure ABC is to KGH, ſo is the parallelogram BE to the parallelogram EF [g]: and the rectilineal figure ABC is equal to the parallelogram BE; therefore the rectilineal figure KGH is equal [h] to the parallelogram EF: but EF is equal to the figure D; wherefore alſo KGH is equal to D; and it is ſimilar to ABC. Therefore the rectilineal figure KGH has been deſcribed ſimilar to the figure ABC, and equal to D. Which was to be done.

g 11. 5.
h 14. 5.

PROP. XXVI. THEOR.

IF two ſimilar parallelograms have a common angle, and be ſimilarly ſituated, they are about the ſame diameter.

Let the parallelograms ABCD, AEFG be ſimilar and ſimilarly ſituated, and have the angle DAB common. ABCD and AEFG are about the ſame diameter.



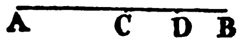
For, if not, let, if poſſible, the parallelogram BD have its diameter AHC in a different ſtraight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC: therefore the parallelograms ABCD, AKHG being about the ſame diameter, they are ſimilar to one another [a]: wherefore, as DA to AB, ſo is [b] GA to AK: but becauſe ABCD and AEFG are ſimilar parallelograms, as DA is to AB, ſo is GA to AE; therefore [c] as GA to AE, ſo GA to AK; wherefore GA has the ſame ratio to each of the ſtraight lines AE, AK; and conſequently AK is equal [d] to AE, the leſs to the greater, which is impoſſible: therefore ABCD and AKHG are not about the ſame diameter; wherefore ABCD and AEFG muſt be about the ſame diameter. Therefore, if two ſimilar, &c. Q. E. D.

a 24. 6.
b 1. def. 6.
c 11. 5.
d 9. 5.

PROP. XXVII. THEOR.

OF all the rectangles contained by the ſegments of a given ſtraight line, the greateſt is the ſquare which is deſcribed on half the line.

Let AB be a given ſtraight line, which is biſected in C; and let D be any point in it, the ſquare on AC is greater than the rectangle AD, DB.



For ſince the ſtraight line AB is divided into two equal parts in C, and into two unequal parts in D, the rectangle contained by AD and DB, together with the ſquare of CD, is equal to the ſquare of AC [a]. The ſquare of AC is therefore greater than the rectangle AD, DB. Therefore, &c. Q. E. D.

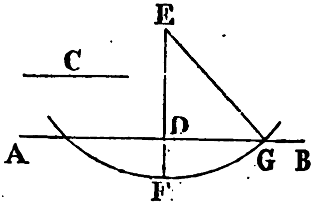
[a] 5. 2.

PROP. XXVIII. PROB.

TO divide a given ſtraight line, ſo that the rectangle contained by its ſegments may be equal to a given ſpace; but that ſpace muſt not be greater than the ſquare of half the given line.

Let AB be the given ſtraight line, and let the ſquare upon the given ſtraight line C be the ſpace to which the rectangle contained by the ſegments of AB muſt be equal, and this ſquare, by the determination, is not greater than that upon half the ſtraight line AB.

Biſect AB in D, and if the ſquare upon AD be equal to the ſquare upon C, the thing required is done: But if it be not equal to it, AD muſt be greater than C, according to the determination: Draw DE at right angles to AB, and make it equal to C; produce ED to F, ſo that EF be equal to AD or DB, and from the centre E, at the diſtance EF, deſcribe a circle meeting AB in G. Join EG; and becauſe AB is divided equally in D, and unequally in G, the rectangle AG, GB, together with the ſquare of



a 5. 2. DG is equal[a] to the ſquare of DB, that is, of EF or EG; but the ſquares of ED, DG are alſo equal to the ſquare of

b 47. 1. EG[b]; therefore the rectangle AG, GB, together with the ſquare of DG, is equal to the ſquares of ED and DG; take away the ſquare of DG from each of theſe equals; therefore the remaining rectangle AG, GB is equal to the ſquare of ED, that is, of C: Wherefore the given line AB is divided is G, ſo that the rectangle contained by the ſegments AG, GB is equal to the ſquare upon the given ſtraight line C. Which was to be done.

PROP. XXIX. PROB.

TO produce a given ſtraight line, ſo that the rectangle contained by the ſegments between the extremities of the given line and the point to which it is produced, may be equal to a given ſpace.

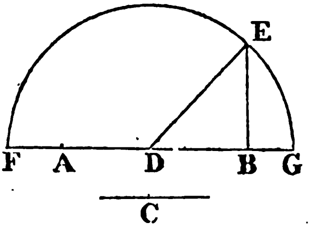
Let AB be the given ſtraight line, and let the ſquare upon the given ſtraight line C be the ſpace to which the rectangle under the ſegments of AB produced, muſt be equal.

Biſect AB in D, and draw BE at right angles to it, ſo that BE be equal to C; and having joined DE, from the centre D at the diſtance DE deſcribe a circle meeting AB produced in

G. And becauſe AB is biſected in D, and produced to G, the rectangle AG, GB together with the ſquare of DB is equal [a] to the ſquare of DG, or of DE, that is, to the ſquares of EB, BD [b]. From each of theſe equals take the ſquare of DB; therefore the remaining rectangle AG, GB is equal to the ſquare of BE, that is, to the ſquare upon C. Wherefore the line AB is produced to G, ſo that the rectangle contained by the ſegments AG, GB of the line produced is equal to the ſquare of C. Which was to be done.

a 6. 2.

b 47. 1.



PROP. XXX. PROB.

TO cut a given ſtraight line in extreme and mean ratio.

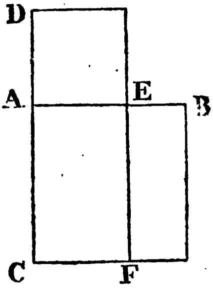
Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Upon AB deſcribe [a] the ſquare BC, and produce CA to D, ſo that the rectangle CD, DA may be equal to the ſquare CB [b]. Take AE equal to AD, and complete the rectangle DF under DC and AE, or under DC and DA. Then, becauſe the rectangle CD, DA is equal to the ſquare CB, the rectangle DF is equal to CB. Take away the common part CE from each, and the remainder FB is equal to the remainder DE. But FB is the rectangle contained by FE and EB, that is, by AB and BE; and DE is the ſquare upon AE; therefore AE is a mean proportional between AB and BE [c], or AB is to AE as AE to EB. But AB is greater than AE;

a 46. 1.

b 29. 6.

c 17. 6.



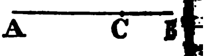
wherefore

e 14. 5. f 3. def. 6. wherefore AE is greater than EB[e]: Therefore the ſtraight line AB is cut in extreme and mean ratio in E[f]. Which was to be done.

Otherwiſe,

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

g 11. 2. h 17. 6. Divide AB in the point C, ſo that the rectangle contained by AB, BC be equal to the ſquare of AC[g]: Then, becauſe the rectangle AB, BC is equal to the ſquare of AC, as BA to AC, ſo is AC to CB[h]: Therefore AB is cut in extreme and mean ratio in C[f]. Which was to be done.



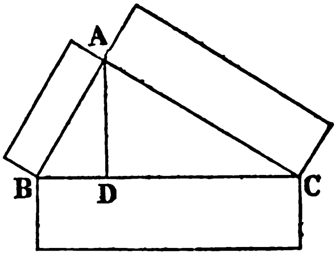
PROP. XXXI. THEOR.

IN right angled triangles, the rectilineal figure deſcribed upon the ſide oppoſite to the right angle, is equal to the ſimilar, and ſimilarly deſcribed figures upon the ſides containing the right angle.

Let ABC be a right angled triangle, having the right angle BAC: The rectilineal figure deſcribed upon BC is equal to the ſimilar, and ſimilarly deſcribed figures upon BA, AC.

a 8. 6. b 4. 6. c 2. Cor. 20. 6. Draw the perpendicular AD; therefore, becauſe in the right angled triangle ABC, AD is drawn from the right angle at A perpendicular to the baſe BC, the triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another[a], and becauſe the triangle ABC is ſimilar to ADB, as CB to BA, ſo is BA to BD[b]; and becauſe theſe three ſtraight lines are proportionals, as the firſt to the third, ſo is the figure upon the firſt to the ſimilar, and ſimilarly deſcribed figure upon the ſecond[c]: Therefore as CB to BD, ſo is the figure upon

CB



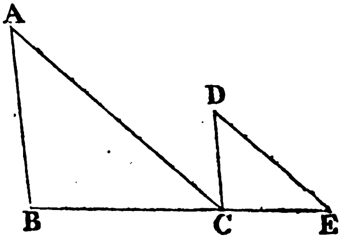
CB to the ſimilar and ſimilarly deſcribed figure upon
BA: and inverſely[d], as d B. 5.
DB to BC, ſo is the figure
upon BA to that upon BC;
or the ſame reaſon, as DC
o CB, ſo is the figure upon
CA to that upon CB.
Wherefore, as BD and DC
ogether to BC, ſo are the
igures upon BA, AC to that upon BC[e]: But BD and DC e 24. 5.
ogether are equal to BC. Therefore the figure deſcribed on
BC is equal[f] to the ſimilar and ſimilarly deſcribed figures f A. 5.
on BA, AC. Wherefore, in right angled triangles, &c.
Q. E. D.

PROP. XXXII. THEOR.

IF two triangles which have two ſides of the one proportional to two ſides of the other, be joined at one angle, ſo as to have their homologous ſides parallel to one another; the remaining ſides ſhall be in a ſtraight line.

Let ABC, DCE be two triangles which have the two ſides BA, AC proportional to the two CD, DE, viz. BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE, BC and CE are in a ſtraight line.

Becauſe AB is parallel to DC, and the ſtraight line AC
meets them, the alternate angles BAC, ACD are equal[a]; for a 29. 1.
the ſame reaſon, the angle CDE is equal to the angle ACD;
wherefore alſo BAC is equal to CDE: And becauſe the triangles ABC, DCE have one angle at A equal to one at D, and the ſides about theſe angles proportionals, viz. BA to AC, as CD to DE, the triangle ABC is equiangular[b] to DCE: Therefore b 6. 6.
the angle ABC is equal to the angle DCE: And the angle BAC was pro-



ved to be equal to ACD: Therefore the whole angle ACE is equal to the two angles ABC, BAC; add the common angle ACB, then the angles ACE, ACB are equal to the angles ABC, BAC, ACB: But ABC, BAC, ACB are equal to two right angles [c]; therefore alſo the angles ACE, ACB are equal to two right angles: And ſince at the point C, in the ſtraight line AC, the two ſtraight lines BC, CE, which are on the oppoſite ſides of it, make the adjacent angles ACE, ACB equal to two right angles; therefore [d] BC and CE are in a ſtraight line. Wherefore, if two triangles, &c. Q. E. D.

c 32. 1.

d 14. 1.

PROP. XXXIII. THEOR.

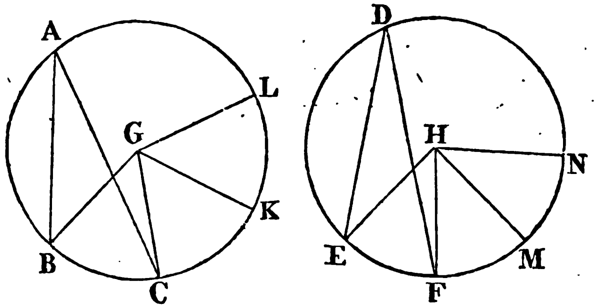
IN equal circles, angles, whether at the centres or circumferences, have the ſame ratio which the circumferences on which they ſtand have to one another: So alſo have the ſectors.

Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF at their circumferences; as the circumference BC to the circumference EF, ſo is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and alſo the ſector BGC to the ſector EHF.

Take any number of circumferences CK, KL, each equal to BC, and any number whatever FM, MN each equal to EF; and join GK, GL, HM, HN. Becauſe the circumferences BC, CK, KL are all equal, the angles BGC, CGK, KGL are alſo all equal [a]: Therefore what multiple ſoever the circumference BL is of the circumference BC, the ſame multiple is the angle BGL of the angle BGC: For the ſame reaſon, whatever multiple the circumference EN is of the circumference EF, the ſame multiple is the angle EHN of the angle EHF: And if the circumference BL be equal to the circumference EN, the angle BGL is alſo equal [a] to the angle EHN; and if the circumference BL be greater than EN, likewiſe the angle BGL is greater than EHN; and if leſs, leſs: There being then four magnitudes, the two circumferences BC, EF, and

a 27. 3.

d the two angles BGC, EHF, of the circumference BC, and the angle BGC, have been taken any equimultiples whater, viz. the circumference BL, and the angle BGL; and of e circumference EF, and of the angle EHF, any equimul-



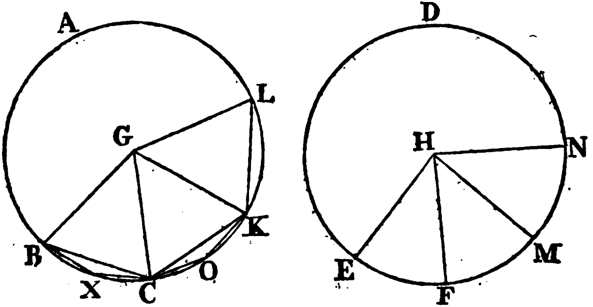
les whatever, viz. the circumference EN, and the angle HN: And it has been proved, that, if the circumference L be greater than EN, the angle BGL is greater than HN; and if equal, equal; and if leſs, leſs: As therefore e circumference BC to the circumference EF, ſo[b] is the gle BGC to the angle EHF: But as the angle BGC is to e angle EHF, ſo is[c] the angle BAC to the angle EDF, for ch is double of each[d]: Therefore, as the circumference BC to EF, ſo is the angle BGC to the angle EHF, and the gle BAC to the angle EDF.

b 5. def. 5.
c 15. 5.
d 20. 3.

Alſo, as the circumference BC to EF, ſo is the ſector BGC the ſector EHF. Join BC, CK, and in the circumferences C, CK take any points X, O, and join BX, XC, CO, OK: hen, becauſe in the triangles GBC, GCK the two ſides BG, C are equal to the two CG, GK, and that they contain eal angles; the baſe BC is equal[e] to the baſe CK, and the angle GBC to the triangle GCK: And becauſe the circumrence BC is equal to the circumference CK, the remaining rt of the whole circumference of the circle ABC, is equal to e remaining part of the whole circumference of the ſame cle: Wherefore the angle BXC is equal to the angle K[a]; and the ſegment BXC is therefore ſimilar to the ſegent COK[f]; and they are upon equal ſtraight lines BC, CK: t ſimilar ſegments of circles upon equal ſtraight lines, are ual[g] to one another: Therefore the ſegment BXC is equal

e 4. 1.
f 9. def. 3.
g 24. 3.

to the ſegment COK: And the triangle BGC is equal to the triangle CGK; therefore the whole, the ſector BGC, is equal to the whole, the ſector CGK: For the ſame reaſon, the ſector KGL is equal to each of the ſectors BGC, CGK: In the ſame manner, the ſectors EHF, FHM, MHN may be proved equal to one another: Therefore, what multiple ſoever the circumference BL is of the circumference BC, the ſame multiple is the ſector BGL of the ſector BGC: For the ſame reaſon, whatever multiple the circumference EN is of EF, the ſame multiple is the ſector EHN of the ſector EHF: And if the circumference BL be equal to EN, the ſector BGL is equal to the ſector EHN; and if the circumference BL be greater than EN, the ſector BGL is greater than the ſector EHN; and if leſs, leſs: Since, then, there are four magnitudes, the two circumferences BC, EF, and the two ſectors BGC, EHF, and of the circumference BC, and ſector BGC, the circumference BL and ſector BGL are any equal multiples whatever; and of the circumference EF, and ſector EHF, the circumference EN and ſector EHN, are any equimultiples whatever; and that it has been proved, if the circumference BL be greater than EN, the ſector BGL is greater than the ſector EHN; and if equal, equal; and if leſs, leſs. Therefore [b], as the circumference BC is to the circumference EF, ſo is the ſector BGC to the ſector EHF. Wherefore, in equal circles, &c. Q. E. D.



[b] 5. def. 5.

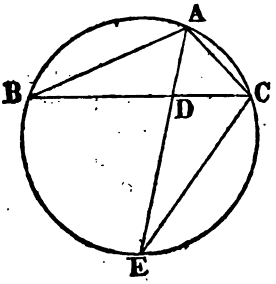
PROP.

PROP. B. THEOR.

IF an angle of a triangle be biſected by a ſtraight line, which likewiſe cuts the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the ſegments of the baſe, together with the ſquare of the ſtraight line biſecting the angle.

Let ABC be a triangle, and let the angle BAC be biſected by the ſtraight line AD; the rectangle BA, AC is equal to the rectangle BD, DC, together with the ſquare of AD.

Deſcribe the circle [a] ACB about the triangle, and produce AD to the circumference in E, and join EC: Then becauſe the angle BAD is equal to the angle CAE, and the angle ABD to the angle [b] AEC, for they are in the ſame ſegment; the triangles ABD, AEC are equiangular to one another: Therefore as BA to AD, ſo is [c] EA to AC, and conſequently the rectangle BA, AC is equal [d] to the rectangle EA, AD, that is [e], to the rectangle ED, DA, together with the ſquare of AD: But the rectangle ED, DA is equal to the rectangle [f] BD, DC. Therefore the rectangle BA, AC is equal to the rectangle BD, DC, together with the ſquare of AD. Wherefore, if an angle, &c. Q. E. D.

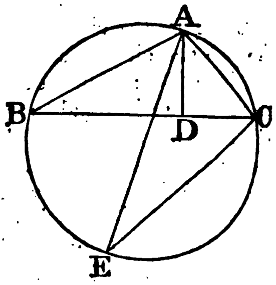


a 5. 4.
b 21. 3.
c 4. 6.
d 16. 6.
e 3. 2.
f 35. 3.

PROP. C. THEOR.

IF from any angle of a triangle a ſtraight line be drawn perpendicular to the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle deſcribed about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the baſe BC; the rectangle BA, AC is equal to the rectangle contained by AD and the diameter of the circle deſcribed about the triangle.



a 5. 4. Deſcribe [a] the circle ACB about the triangle, and draw its diameter AE, and join EC: Becauſe the right angle BDA is equal [b] to the angle ECA in a ſemicircle, and the angle ABD to the angle AEC in the ſame ſegment [c]; the triangles ABD, AEC are equiangular: Therefore, as [d] BA to AD, ſo is EA to AC; and conſequently the rectangle BA, AC is equal [e] to the rectangle EA, AD. If therefore, from an angle, &c. Q. E. D.

b 3. 3.
c 21. 3.
d 4. 6.
e 16. 6.

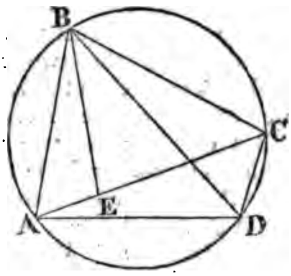
PROP. D. THEOR.

See N. THE rectangle contained by the diagonals of a quadrilateral inſcribed in a circle, is equal to both the rectangles contained by its oppoſite ſides.

Let ABCD be any quadrilateral inſcribed in a circle, and join AC, BD; the rectangle contained by AC, BD is equal to the two rectangles contained by AB, CD, and by AD, BC.

Make the angle ABE equal to the angle DBC; add to each of theſe the common angle EBD, then the angle ABD is equal to the angle EBC: And the angle BDA is equal[a] to the angle BCE, becauſe they are in the ſame ſegment; therefore the triangle ABD is equiangular to the triangle BCE: Wherefore[b] as BC is to CE, ſo is BD to DA; and conſequently the rectangle BC, AD is equal[c] to the rectangle BD, CE: Again, becauſe the angle ABE is equal to the angle DBC, and the angle[a] BAE to the angle BDC, the triangle ABE is equiangular to the triangle BCD: As therefore BA to AE, ſo is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE: But the rectangle BC, AD has been ſhewn equal to the rectangle BD, CE; therefore the whole rectangle AC, BD[d] is equal to the rectangle AB, DC, together with the rectangle AD, BC. Therefore the rectangle, &c. Q. E. D.

a 21. 3.
b 4. 6.
c 16. 6.
d 1. 2.

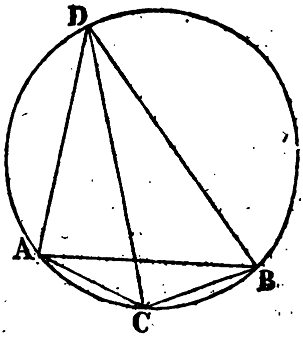


PROP. E. THEOR.

IF a ſegment of a circle be biſected, and from the extremities of the baſe of the ſegment, and from the point of biſection ſtraight lines be inflected to any point in the circumference, the ſum of the two lines drawn from the extremities of the baſe will have to the line drawn from the point of biſection the ſame ratio which the baſe of the ſegment has to the baſe of half the ſegment.

Let ABD be a circle, of which AB is a ſegment biſected in C, and from A, C and B to D, any point whatever in the circumference, let AD, CD, BD be drawn; the ſum of the two lines AD and DB has to DC the ſame ratio that BA has to AC.

For ſince ACBD is a quadrilateral inſcribed in a circle, of which the diagonals are AB and CD, the rectangles AD, CB, and DB, AC are together equal to the rectangle[a] AB, CD. But the rectangle AD, BC is equal to the rectangle AD, AC, becauſe BC is equal to AC; and therefore the two rectangles AD, AC and BD, AC are equal to the rectangle AB, CD. But the two rectangles AD, AC, and BD, AC, are the rectangle contained by AC, and the ſum of the lines AD and DB[b]. Wherefore the rectangle contained by AC and the ſum of the lines AD, DB is equal to the rectangle AB, CD; and becauſe the ſides of equal rectangles are reciprocally proportional[c], the ſum of AD and DB is to DC as AB to AC. Q. E. D.



a D. 6.
b 1. 2.
c 14. 6.

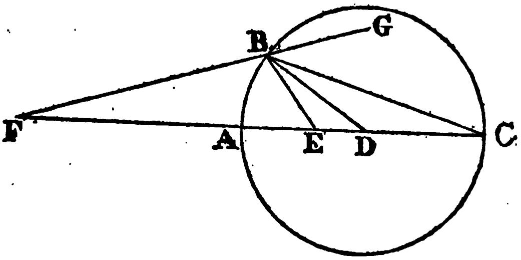
PROP. F. THEOR.

IF two points be taken in the diameter of a circle, ſuch that the rectangle contained by the ſegments intercepted between them and the centre of the circle be equal to the ſquare of the ſemidiameter; and if from theſe points two ſtraight lines be inflected to any point whatſoever in the circumference of the circle, the ratio of the lines inflected will be the ſame with the ratio of the ſegments intercepted between the two firſt mentioned points and the circumference of the circle.

Let ABC be a circle, of which the centre is D, and in DA produced, let the points E and F be ſuch that the rectangle

ED, DF is equal to the ſquare of AD; from E and F to any
point B in the circumference, let EB, FB be drawn;
FB : BE : : FA : AE.

Join BD, and becauſe the rectangle FD, DE is equal to the
ſquare of AD, that is, of DB, FD : DB : : DB : DE[a]. The a 17. 6.



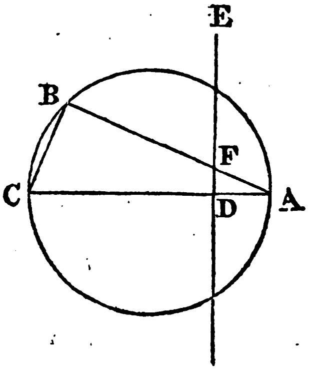
two triangles, FDB, BDE have therefore the ſides proportional
that are about the common angle D; therefore they
are equiangular[b], the angle DEB being equal to the angle b 6. 6.
DBF, and DBE to DFB. Now, ſince the ſides about theſe
equal angles are alſo proportional[c], FB : BD : : BE : ED, and c 4. 6.
alternately[d], FB : BE : : BD : ED, or FB : BE : : AD : DE. d 16. 5.
But becauſe FD : DA : : DA : DE, by diviſion[e], FA : DA : : e 17. 5.
AE : ED, and alternately[e], FA : AE : : DA : ED. Now,
it has been ſhewn that FB : BE : : AD : DE, therefore[f] f 11. 5.
FB : BE : : FA : AE. Therefore, &c. Q. E. D.

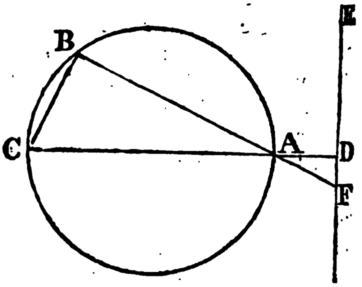
COR. If AB be drawn, becauſe FB : BE : : FA : AE, the
angle FBE is biſected[g] by AB. Alſo, ſince FD : DC : : g 3. 6.
DC : DE, by compoſition[h], FC : DC : : CE : ED, and ſince it h 18. 5.
has been ſhewn that FA : AD (DC) : : AE : ED, therefore,
ex æquo, FA : AE : : FC : CE. But FB : BE : : FA : AE,
therefore, FB : BE : : FC : CE[f]; ſo that if FB be produced
to G, and if BC be drawn, the angle EBG is biſected by the k A. 6.
line BC[k].

PROP. G. THEOR.

IF from the extremity of the diameter of a circle a ſtraight line be drawn in the circle, and if either within the circle or produced without it, it meet a line perpendicular to the ſame diameter, the rectangle contained by the ſtraight line drawn in the circle, and the ſegment of it intercepted between the extremity of the diameter and the perpendicular, is equal to the rectangle contained by the diameter and the ſegment of it cut off by the perpendicular.

Let ABC be a circle, of which AC is a diameter, let DE be perpendicular to the diameter AC, and let AB meet DE in F; the rectangle BA, AF is equal to the rectangle CA, AD. Join BC, and becauſe ABC is an angle in a ſemicircle,



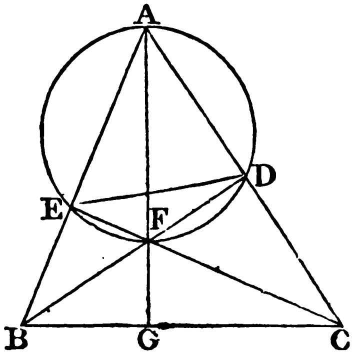


a 31. 3. it is a right angle[a]: Now, the angle ADF is alſo a right
b Hyp. angle[b]; and the angle BAC is either the ſame with DAF, or vertical to it; therefore the triangles ABC, ADF are
c 4. 6. equiangular, and BA : AC : : AD : AF[c]; therefore alſo the rectangle BA, AF, contained by the extremes, is equal to
d 16. 6. the rectangle AC, AD contained by the means[d]. If therefore, &c. Q. E. D.

PROP. H. THEOR.

HE perpendiculars drawn from the three
angles of any triangle to the oppoſite ſides in-
t one another in the ſame point.

ABC be a triangle, BD and CE two perpendiculars
cting one another in F ; let AF be joined, and, produced
ſſary, let it meet BC in G, AG is perpendicular to BC.
DE, and about the triangle AEF let a circle be de-
, AEF ; then, becauſe AEF is a right angle, the
deſcribed about the
e AEF will have
r its diameter[a]. In a 31. 3.
e manner, the circle
ed about the triangle
has AF for its di-
; therefore the points
F and D are in the
ference of the ſame
But becauſe the
FB is equal to the
FC [b], and alſo the b 15. 1.
BEF to the angle
being both right
the triangles BEF



F are equiangular, and therefore BF:EF :: CF:FD,[c] c 4. 6.
nately[d], BF : FC :: EF : FD. Since, then, the ſides d 16. 5.
he equal angles BFC, EFD are proportionals, the tri-
BFC, EFD are alſo equiangular[e] ; wherefore the angle e 6. 6.
s equal to the angle EDF. But EDF is equal to
becauſe they are angles in the ſame ſegment[f] ; there- f 21. 3.
e angle EAF is equal to the angle FCG : Now, the
AEF, CFG are alſo equal, becauſe they are vertical
, therefore the remaining angles AEF, FGC are alſo
: But AEF is a right angle, therefore FGC is a right g 32. 1.
nd AG is perpendicular to BC. Q. E. D.

ELE-

ELEMENTS

OF

GEOMETRY.

BOOK VII.

DEFINITIONS.

I.

Book VII.

A ſtraight line is perpendicular, or at right angles to a plane, when it makes right angles with every ſtraight line meeting it in that plane.

II.

A plane is perpendicular to a plane, when the ſtraight lines drawn in one of the planes perpendicularly to the common ſection of the two planes, are perpendicular to the other plane.

III.

The inclination of a ſtraight line to a plane is the acute angle contained by that ſtraight line and another drawn from the point in which the firſt line meets the plane, to the point in which a perpendicular to the plane drawn fro any point of the firſt line above the plane, meets the ſam plane.

IV.

The inclination of a plane to a plane is the acute angle co tained by two ſtraight lines drawn from any the ſame poi of their common ſection at right angles to it, one upon o plane, and the other upon the other plane.

V.

Two planes are ſaid to have the ſame, or a like inclination one another, which two other planes have, when the ſ angles of inclination are equal to one another.

VI.

Parallel planes are ſuch as do not meet one another thou produced.

VII.

A Solid is that which has length, breadth, and thickneſs.

VIII.

See N. A ſolid angle is that which is made by the meeting of m than two plane angles, which are not in the ſame plane, one point.

IX.

See N. Similar ſolid figures are ſuch as are contained by the ſa number of ſimilar planes ſimilarly ſituated, and having l inclinations to one another.

X.

A pyramid is a ſolid figure contained by planes that are c ſtituted betwixt one plane and a point above it in wh they meet.

XI.

A priſm is a ſolid figure contained by plane figures, of which two that are oppoſite are equal, ſimilar, and parallel to one another; and the others parallelograms.

XII.

A parallelopiped is a ſolid figure contained by ſix quadrilateral figures, whereof every oppoſite two are parallel.

XIII.

A cube is a ſolid figure contained by ſix equal ſquares.

XIV.

A ſphere is a ſolid figure deſcribed by the revolution of a ſemicircle about a diameter, which remains unmoved.

XV.

The axis of a ſphere is the fixed ſtraight line about which the ſemicircle revolves.

XVI.

The centre of a ſphere is the ſame with that of the ſemicircle.

XVII.

The diameter of a ſphere is any ſtraight line which paſſes through the centre, and is terminated both ways by the ſuperficies of the ſphere.

XVIII.

A cone is a ſolid figure deſcribed by the revolution of a right angled triangle about one of the ſides containing the right angle, which ſide remains fixed.

XIX.

The axis of a cone is the fixed ſtraight line about which the triangle revolves.

XX.

The baſe of a cone is the circle deſcribed by that ſide taining the right angle, which revolves.

XX.

A cylinder is a ſolid figure deſcribed by the revolution right angled parallelogram about one of its ſides whic mains fixed.

XXI.

The axis of a cylinder is the fixed ſtraight line about w the parallelogram revolves.

XXII.

The baſes of a cylinder are the circles deſcribed by the revolving oppoſite ſides of the parallelogram.

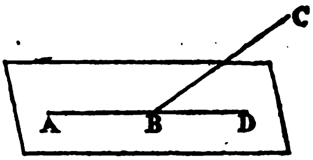
XXIII.

Similar cones and cylinders are thoſe which have their and the diameters of their baſes proportionals.

PROP. I. THEOR.

ONE part of a ſtraight line cannot be in a plane and another part above it. See N.

If it be poſſible, let AB, part of the ſtraight line ABC, be n the plane, and the part BC above it: and ſince the ſtraight ine AB is in the plane, it can e produced in that plane: let t be produced to D: Then BC and ABD are two raight lines, and they have he common ſegment AB, hich is impoſſible [b]. There-ore ABC is not a ſtraight line. Wherefore one part, &c. Q. E. D.



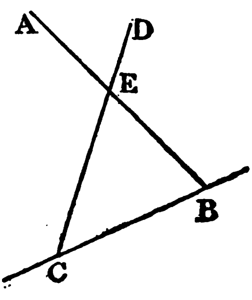
b cor. def. 4. 1.

PROP. II. THEOR.

ANY three ſtraight lines which meet one another, not in the ſame point, are in one plane.

Let the three ſtraight lines AB, CD, CB, meet one anoher in the points B, C and E; AB, CD, CB are in one lane.

Let any plane paſs through the raight line EB, and let the plane e turned about EB, produced, if eceſſary, until it paſs through the oint C: Then, becauſe the points , C are in this plane, the ſtraight ine EC is in it [a]: for the ſame reaon, the ſtraight line BC is in the ame; and, by the hypotheſis, EB is n it: therefore the three ſtraight ines EC, CB, BE are in one plane: ut the whole of the lines DC, AB, nd BC produced, are in the ſame plane with the parts of them



a 5. def. 1.

EC, EB, BC [b]. Therefore AB, CD, CB, are all in one plane. Wherefore, &c. Q. E. D.

[b] 1. 7.

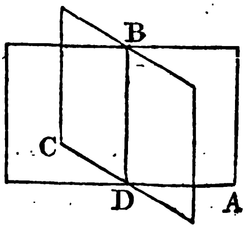
Cor. It is manifeſt, that any two ſtraight lines which cut one another are in one plane.

PROP. III. THEOR.

IF two planes cut one another, their common ſection is a ſtraight line.

Let two planes AB, BC cut one another, and let B and D be two points in the line of their common ſection. From B to D draw the ſtraight line BD; and becauſe the points B and D are in the plane AB, the ſtraight line BD is in that plane [a]: for the ſame reaſon it is in the plane CB; the ſtraight line BD is therefore common to the planes AB and BC, or it is the common ſection of theſe planes. Therefore, &c. Q. E. D.

[a] 5. def 1.



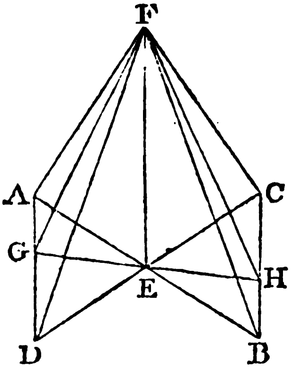
PROP. IV. THEOR.

IF a ſtraight line ſtand at right angles to each of two ſtraight lines in the point of their interſection, it will alſo be at right angles to the plane in which theſe lines are.

Let the ſtraight line EF ſtand at right angles to each of the ſtraight lines AB, CD in E, the point of their interſection: EF is alſo at right angles to the plane paſſing through AB, CD.

Take the ſtraight lines AE, EB, CE, ED all equal to one another; and through E draw, in the plane in which are AB, CD, CD any ſtraight line GEH; and join AD, CB; then, from any point F in EF, draw FA, FG, FD, FC, FH, FB: and becauſe the two ſtraight lines, AE, ED are equal to the two BE, EC, and that they contain equal angles [a] AED, BEC, the baſe AD is equal [b] to the baſe BC, and the angle DAE to the angle EBC: and the angle AEG is equal to the angle BEH [a]; therefore the triangles AEG, BEH have two angles of one equal to two angles of the other, each to each, and the ſides AE, EB, adjacent to the equal angles, equal to one another; wherefore they ſhall have their other ſides equal [c]: GE is therefore equal to EH, and AG to BH: and becauſe AE is equal to EB, and FE common and at right angles to them, the baſe AF is equal [b] to the baſe FB; for the ſame reaſon, CF is equal to FD: and becauſe AD is equal to BC, and AF to FB; and becauſe the baſe DF was proved equal to the baſe FC; therefore the angle FAD is equal [d] to the angle FBC. Again, it was proved that GA is equal to BH, and AF to FB; and the angle FAG has alſo been proved equal to the angle FBH; therefore the baſe GF is equal [b] to the baſe FH: And becauſe GE is equal to EH, as was already proved, and EF is common to the triangles GFE, HFE; and alſo the baſe GF equal to the baſe FH; the angle GEF is equal [d] to the angle HEF; and conſequently each of theſe angles is a right [e] angle. Therefore FE makes right angles with GH, that is with any ſtraight line drawn through E in the plane paſſing through AB, DC; therefore it makes right angles with every ſtraight line which meets it in that plane. But a ſtraight line is at right angles to a plane when it makes right angles with every ſtraight line which meets it in that plane [f]: therefore EF is at right angles to the plane in which are AB, CD. Wherefore, if a ſtraight line, &c. Q. E. D.

[a] 5. 1.
[b] 4. 1.
[c] 26. 1.
[b] 4. 1.
[d] 8. 1.
[e] 7. def. 1.
[f] 1. def. 7.

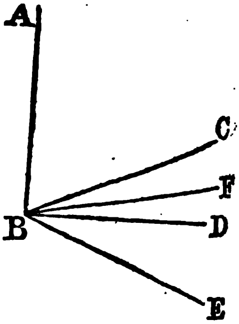


PROP. V. THEOR.

IF three ſtraight lines meet all in one point, and a ſtraight line ſtand at right angles to each of them in that point: theſe three ſtraight lines are in one and the ſame plane.

Let the ſtraight line AB ſtand at right angles to each of the ſtraight lines BC, BD, BE, in B, the point where they meet; BC, BD, BE are in one and the ſame plane.

If not, let, if it be poſſible, BD and BE be in one plane, and BC be above it; and let a plane paſs through AB, BC, the common ſection of which with the plane, in which BD and
a 3. 7. BE are, ſhall be a ſtraight [a] line; let this be BF: therefore the three ſtraight lines AB, BC, BF are all in one plane, viz. that which paſſes through AB, BC; and becauſe AB ſtands at right angles to each of the ſtraight lines BD, BE, it is al-
b 1. 11. ſo at right angles [b] to the plane paſſing through them; and
c 1.def.7. therefore makes right angles [c] with every ſtraight line meeting it in that plane; but BF which is in that plane meets it: therefore the angle ABF is a right angle; but the angle ABC, by the hypotheſis, is alſo a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the ſame plane, which is impoſſible: therefore the ſtraight line BC is not above the plane in which are BD and BE: Wherefore the three ſtraight lines BC, BD, BE are in one and the ſame plane. Therefore, if three ſtraight lines, &c. Q. E. D.



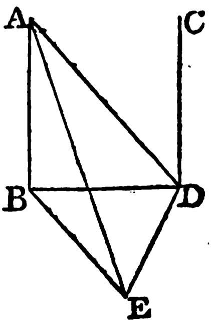
PROP.

PROP. VI. THEOR.

F two ſtraight lines be at right angles to the ſame plane, they ſhall be parallel to one another.

Let the ſtraight lines AB, CD be at right angles to the ſame ne; AB is parallel to CD.

Let them meet the plane in the points B, D, and draw the ight line BD, to which draw DE at right angles, in the ne plane; and make DE equal to 3, and join BE, AE, AD. Then, ·auſe AB is perpendicular to the ne, it ſhall make right [a] angles :h every ſtraight line which meets it, l is in that plane: but BD, BE, ich are in that plane, do each of m meet AB. Therefore each of angles ABD, ABE is a right angle: the ſame reaſon, each of the angles)B, CDE is a right angle: and beſe AB is equal to DE, and BD nmon, the two ſides AB, BD, are ıal to the two ED, DB; and ·y contain right angles; therefore the baſe AD is equal [b] to baſe BE. Again, becauſe AB is equal to DE, and BE to D, and the baſe AE common to the triangles ABE, EDA; angle ABE is equal [c] to the angle EDA: but ABE is a ht angle; therefore EDA is alſo a right angle, and ED perıdicular to DA: but it is alſo perpendicular to each of the o BD, DC: Wherefore ED is at right angles to each of three ſtraight lines BD, DA, DC in the point in which y meet: Therefore theſe three ſtraight lines are all in the ne plane [d]: but AB is in the plane in which are BD, DA, auſe any three ſtraight lines which meet one another are in plane [e]: Therefore AB, BD, DC are in one plane; and h of the angles ABD, BDC is a right angle; therefore AB parallel [f] to CD. Wherefore, if two ſtraight lines, &c. E. D.



a 1. def. 7.

b 4. 1.

c 8. 1.

d 5. 7.

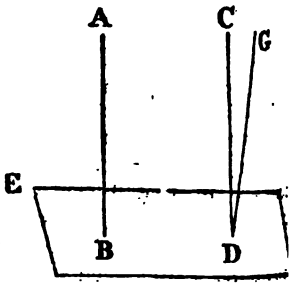
e 2. 7.

f 28. 1.

Book VII.

PROP. VII. THEOR.

IF two ſtraight lines be parallel, and one of th is at right angles to a plane; the other alſo ſh be at right angles to the ſame plane.



Let AB, CD be two parallel ſtraight lines, and let o them AB be at right angles to a plane; the other CD is at right angles to the ſame plane.

For, if CD be not perpendicular to the plane to which AB is perpendicular, let DG be perpendicular to it. Then [a] DG is parallel to AB: DG and DC therefore are both parallel to AB, and are drawn through the ſame point D, which is impoſſible [b].

[a] 6. 7.

[b] 11. Ax. 1.

PROP. VIII. THEOR.

TWO ſtraight lines which are each of them rallel to the ſame ſtraight line, though not in the ſame plane with it, are parallel to one anot

Let AB, CD be each of them parallel to EF, and n the ſame plane with it; AB ſhall be parallel to CD.

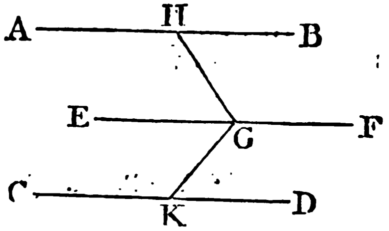
In EF take any point G, from which draw, in the paſſing through EF, AB, the ſtraight line GH at angles to EF; and in the plane paſſing through EF, draw GK at right angles to the ſame EF. And becauſ is perpendicular both to GH and GK, EF is perpendic

[a] 4. 7.

to the plane HGK paſſing through them: and EF is parallel to AB; therefore AB is at right angles [b] to the plane HGK. For the ſame reaſon, CD is likewiſe at right angles to the plane HGK. Therefore AB, CD are each of them at right angles to the plane HGK. But if two ſtraight lines are at right angles to the ſame plane, they are parallel [c] to one another. Therefore AB is parallel to CD. Wherefore two ſtraight lines, &c. Q. E. D.

b 7. 7.

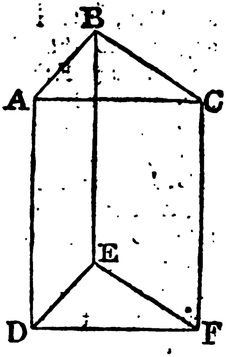
c 6. 7.



PROP. IX. THEOR.

IF two ſtraight lines meeting one another be parallel to two others that meet one another, though not in the ſame plane with the firſt two; the firſt two and the other two ſhall contain equal angles.

Let the two ſtraight lines AB, BC which meet one another be parallel to the two ſtraight lines DE, EF that meet one another, and are not in the ſame plane with AB, BC. The angle ABC is equal to the angle DEF.



Take BA, BC, ED, EF all equal to one another; and join AD, CF, BE, AC, DF: becauſe BA is equal and parallel to ED, therefore AD is [a] both equal and parallel to BE. For the ſame reaſon, CF is equal and parallel to BE. Therefore AD and CF are each of them equal and parallel to BE. But ſtraight lines that are parallel to the ſame ſtraight line, though not in the ſame plane with it, are parallel [b] to one another. Therefore AD is parallel to CF; and

a 33. 1.

b 8. 7.

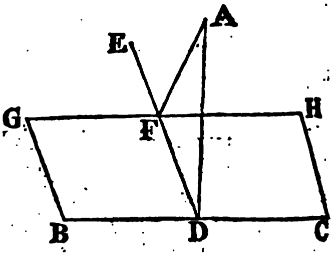
and it is equal to it, and AC, DF join them towards the ſame
parts; and therefore a AC is equal and parallel to DF. And
becauſe AB, BC are equal to DE, EF, and the baſe AC to
c 8. 1. the baſe DF; the angle ABC is equal c to the angle DEF.
Therefore, if two ſtraight lines, &c. Q. E. D.

PROP. X. THEOR.

TO draw a ſtraight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a ſtraight line perpendicular to the plane BH.

In the plane draw any ſtraight line BC, and from the point
a 12. 1. A draw a AD perpendicular to BC. If then AD be alſo
perpendicular to the plane BH, the thing required is already
done; but if it be not, from
b 11. 1. the point D draw b, in the
plane BH, the ſtraight line
DE at right angles to BC;
and from the point A draw
AF perpendicular to DE;
c 31. 1. and through F draw c GH
parallel to BC: and becauſe
BC is at right angles to ED
and DA, BC is at right
d 4. 7. angles d to the plane paſ-
ſing through ED, DA. And GH is parallel to BC; but, if
two ſtraight lines be parallel, one of which is at right angles
e 7. 7. to a plane, the other ſhall be at right e angles to the ſame
plane; wherefore GH is at right angles to the plane through
f 1. def. 7. ED, DA, and is perpendicular f to every ſtraight line meet-
ing it in that plane. But AF, which is in the plane through
ED, DA, meets it: Therefore GH is perpendicular to
AF; and conſequently AF is perpendicular to GH;
and AF is alſo perpendicular to DE: Therefore AF is per-
pendicular to each of the ſtraight lines GH, DE. But if
a ſtraight line ſtands at right angles to each of two ſtraight
lines in the point of their interſection, it ſhall alſo be at right



angles

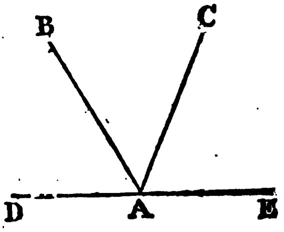
les to the plane paſſing through them[d]. And the plane paſ- g through ED, GH is the plane BH; therefore AF is per- idicular to the plane BH; ſo that, from the given point above the plane BH, the ſtraight line AF is drawn per- idicular to that plane. Which was to be done.

Cor. If it be required from a point C in a plane to ct a perpendicular to that plane, take a point A above the ne, and draw AF perpendicular to the plane, then, if from a line be drawn parallel to AF, it will be the perpendicu- required; for being parallel to AF it will be perpendicu- to the ſame plane to which AF is perpendicular[e].

PROP. XI. THEOR.

ROM the ſame point in a given plane, there cannot be two ſtraight lines at right angles to e plane, upon the ſame ſide of it: And there can but one perpendicular to a plane from a point ove the plane.

For, if it be poſſible, let the two ſtraight lines AC, AB be at ht angles to a given plane from the ſame point A in the ne, and upon the ſame ſide of it; and let a plane paſs ough BA, AC; the common ſection of this with the given ne is a ſtraight[a] line paſſing through A: Let DAE be ir common ſection: Therefore the ſtraight lines AB, AC, AE are in one plane; And becauſe CA is at right angles the given plane, it ſhall make ht angles with every ſtraight e meeting it in that plane. But AE, which is in that plane, ets CA; therefore CAE is a ht angle. For the ſame reaſon AE is a right angle. Where- e the angle CAE is equal to angle BAE; and they are in plane, which is impoſſible. ſo, from a point above a plane, there can be but one per- idicular to that plane; for, if there could be two, they would



[a] 3. 7.

would be parallel [b] to one another, which is abſurd. Therefore, from the ſame point, &c. Q. E. D.

b 6. 7.

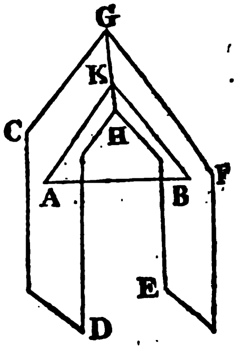
PROP. XII. THEOR.

PLANES to which the ſame ſtraight line is perpendicular, are parallel to one another.

Let the ſtraight line AB be perpendicular to each of the planes CD, EF; theſe planes are parallel to one another.

If not, they ſhall meet one another when produced; let them meet; their common ſection ſhall be a ſtraight line GH, in which take any point K, and join AK, BK: Then, becauſe AB is perpendicular to the plane EF, it is perpendicular [a] to the ſtraight line BK which is in that plane. Therefore ABK is a right angle. For the ſame reaſon, BAK is a right angle; wherefore the two angles ABK, BAK of the triangle ABK are equal to two right angles, which is impoſſible [b]: Therefore the planes CD, EF, though produced, do not meet one another; that is, they are parallel [c]. Therefore planes, &c. Q. E. D.

a 3. def. 7.
b 17. 1.
c 6. def. 7.

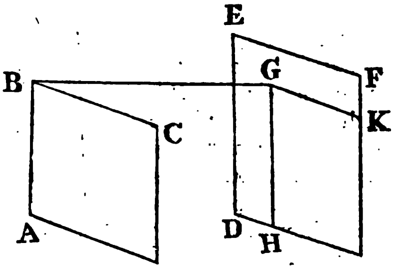


PROP. XIII. THEOR.

IF two ſtraight lines meeting one another, be parallel to two ſtraight lines which meet one another, but are not in the ſame plane with the firſt two; the plane which paſſes through theſe is parallel to the plane paſſing through the others.

Let

AB, BC, two ſtraight lines meeting one another, be
l to DE, EF that meet one another, but are not in the
lane with AB, BC: The planes through AB, BC, and
F ſhall not meet, though produced.
n the point B draw BG perpendicular[a] to the plane a 10. 7.
paſſes through DE, EF, and let it meet that plane in
d through G draw GH parallel[b] to ED, and GK pa- b 31. 1.
o EF: And becauſe BG is perpendicular to the plane
h DE, EF, it
ake right angles
every ſtraight
eeting it in that
. But the ſtraight c 1. def. 7.
GH, GK in that
meet it: There-
ch of the angles
BGK is a right
And becauſe
parallel[d] to GH d 8. 7.
ach of them is
l to DE,) the



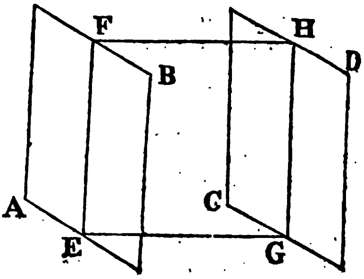
GBA, BGH are together equal[e] to two right angles: e 29. 1.
BGH is a right angle; therefore alſo GBA is a
angle, and GB perpendicular to BA: For the ſame
, GB is perpendicular to BC: Since therefore the
t line GB ſtands at right angles to the two ſtraight lines
BC, that cut one another in B; GB is perpendicular[f] f 4. 7.
plane through BA, BC: And it is perpendicular to the
through DE, EF; therefore BG is perpendicular to
f the planes through AB, BC, and DE, EF: But planes
ch the ſame ſtraight line is perpendicular, are parallel[g] g 12. 7.
another: Therefore the plane through AB, BC is pa-
to the plane through DE, EF. Wherefore, if two
t lines, &c. Q. E. D.

PROP.

PROP. XIV. THEOR.

IF two parallel planes be cut by another plane, their common ſections with it are parallels.

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common ſections with it be EF, GH; EF is parallel to GH.



For the ſtraight lines EF and GH are in the ſame plane, viz. EFHG, which cuts the planes AB and CD; and they do not meet though produced; for the planes in which they are do not meet; therefore EF and GH are parallel [a]. Q. E. D.

a 30. def. 1.

PROP. XV. THEOR.

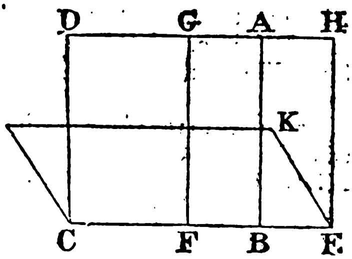
IF two parallel planes be cut by a third plane, they have the ſame inclination to that plane.

Let AB and CD be two parallel planes, and EH a third plane cutting them: The planes AB and CD are equally inclined to EH.

Let the ſtraight lines EF and GH be the common ſection of the plane EH with the two planes AB and CD; and from K any point in EF, draw in the plane EH the ſtraight line KM at right angles to EF, and let it meet GH in L; draw alſo KN at right angles to EF in the plane AB: and through the ſtraight lines KM, KN, let a plane be made to paſs cutting the plane CD in the line LO. And becauſe EF and GH are the common ſections of the plane EH with the two

Let the ſtraight line AB be at right angles to a plane CK; every plane which paſſes through AB ſhall be at right angles to the plane CK.

Let any plane DE paſs through AB, and let CE be the common ſection of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE at right angles to CE: And becauſe AB is perpendicular to the plane CK, therefore it is alſo perpendicular to every ſtraight line meeting it in that plane [a]; and conſequently it is perpendicular to CE: Wherefore ABF is a right angle; but GFB is likewiſe a right angle; therefore AB is parallel [b] to FG. And AB is at right angles to the plane CK; therefore FG is alſo at right angles to the ſame plane [c]. But one plane is at right angles to another plane when the ſtraight lines drawn in one of the planes, at right angles to their common ſection, are alſo at right angles to the other plane [d]; and any ſtraight line FG in the plane DE, which is at right angles to CE the common ſection of the planes, has been proved to be perpendicular to the other plane CK; therefore the plane DE is at right angles to the plane CK. In like manner, it may be proved that all the planes which paſs through AB are at right angles to the plane CK. Therefore, if a ſtraight line, &c. Q. E. D.



a 1. def. 7.
b 28. 1.
c 7. 7.
d 2. def. 7.

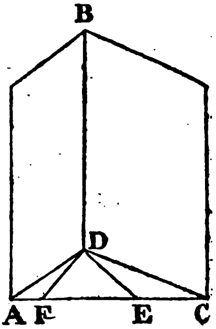
PROP. XVIII. THEOR.

IF two planes cutting one another be each of them perpendicular to a third plane, their common ſection ſhall be perpendicular to the ſame plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common ſection of the firſt two; BD is perpendicular to the plane ADC.

Book XI. From D in the plane ADC, draw DE perpendicular to AD, and DF to DC. Becauſe DE is perpendicular to AD, the common ſection of the planes AB and ADC; and becauſe the plane AB is at right angles to ADC, DE is at right angles to the plane AB [a], and therefore alſo to the ſtraight line BD in that plane [b]. For the ſame reaſon, DF is at right angles to DB. Since BD is therefore at right angles to both the lines DE and DF, it is at right angles to the plane in which DE and DF are, that is, to the plane ADC [c]. Wherefore, &c. Q. E. D.

a 2. def. 7.
b 1. def. 7.
c 4. 7.



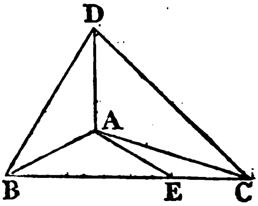
PROP. XIX. THEOR.

IF a ſolid angle be contained by three plane angles, any two of them are greater than the third.

Let the ſolid angle at A be contained by the three plane angles BAC, CAD, DAB. Any two of them are greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. But if, they are not, let BAC be that angle which is not leſs than either of the other two, and is greater than one of them DAB; and at the point A in the ſtraight line AB, make, in the plane which paſſes through BA, AC, the angle BAE equal [a] to the angle DAB; and make AE equal to AD, and through E draw BEC cutting AB, AC in the points B, C, and join DB, DC. And becauſe DA is equal to AE, and AB is common to the two triangles ABD, ABE, and alſo the angle DAB equal to the angle EAB; therefore

a 23. 1.



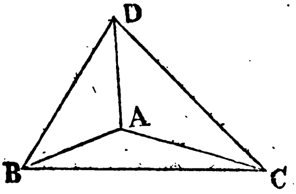
fore the baſe DB is equal [b] to the baſe BE. And be-
cauſe BD, DC are greater [c] than CB, and one of them BD
has been proved equal to BE a part of CB, therefore the o- b 4. 1.
ther DC is greater than the remaining part EC. And be- c 20. 1.
cauſe DA is equal to AE, and AC common, but the baſe
DC greater than the baſe EC; therefore the angle DAC is
greater [d] than the angle EAC; and, by the conſtruction, the d 25. 1.
angle DAB is equal to the angle BAE; wherefore the angles
DAB, DAC are together greater than BAE, EAC, that is,
than the angle BAC. But BAC is not leſs than either of
the angles DAB, DAC; therefore BAC, with either of
them, is greater than the other. Wherefore, if a ſolid angle,
&c. Q. E. D.

PROP. XX. THEOR.

EVERY ſolid angle is contained by plane angles which together are leſs than four right angles.

Firſt, Let the ſolid angle at A be contained by three plane angles BAC, CAD, DAB. Theſe three together are leſs than four right angles.

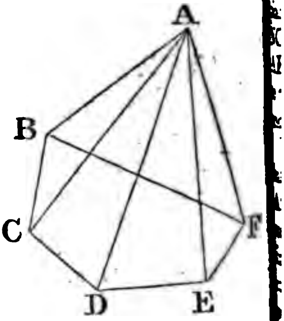
Take in each of the ſtraight lines AB, AC, AD any
points B, C, D, and join BC, CD, DB: then, becauſe the ſo-
lid angle at B is contained by the three plane angles CBA,
ABD, DBC, any two of them are greater [a] than the third; a 19. 7.
therefore the angles CBA, ABD are greater than the angle
DBC: for the ſame reaſon, the angles BCA, ACD are great-
er than the angle DCB; and
the angles CDA, ADB greater
than BDC: Wherefore the ſix
angles CBA, ABD, BCA,
ACD, CDA, ADB are great-
er than the three angles DBC,
BCD, CDB: but the three
angles DBC, BCD, CDB are
equal to two right angles [b]; b 23. 1.
Therefore the ſix angles CBA, ABD, BCA, ACD, CDA,
ADB are greater than two right angles: and becauſe the



Book VII. three angles of each of the triangles ABC, ACD, ADB are equal to two right angles, therefore the nine angles of these three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD are equal to six right angles: Of these, the six angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles: therefore the remaining three angles BAC, DAC, BAD, which contain the solid angle at A, are less than four right angles.

Next, Let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; these together are less than four right angles.

Let the planes in which the angles are, be cut by a plane, and let the common sections of it with those planes be BC, CD, DE, EF, FB: and because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater [a] than the third, the angles CBA, ABF are greater than the angle FBC: for the same reason, the two plane angles at each of the points C, D, E, F, viz. the angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the polygon BCDEF: therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon: and because all the angles of the triangles are together equal to twice as many right angles as there are triangles [b]; that is, as there are sides in the polygon BCDEF; and because all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles as there are sides in the polygon [c]; therefore all the angles of the triangles are equal to all the angles of the polygon together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the polygon, as has been proved. Wherefore, the remaining angles of the triangles, viz. those at the vertex, which contain the solid



a 19. 7.

b 32. 1.

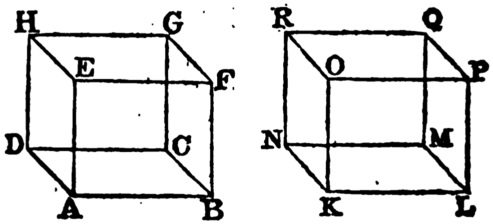
c 1. cor. 32. 1.

angle at A, are leſs than four right angles. Therefore every ſolid angle, &c. Q. E. D.

PROP. XXI. THEOR.

IF two ſolids be contained by the ſame number of equal and ſimilar planes, ſimilarly ſituated, and if the inclination of any two contiguous planes in the one ſolid be the ſame with the inclination of the two equal, and ſimilarly ſituated planes in the other, the ſolids themſelves are equal and ſimilar.

Let AG and KQ be two ſolids contained by the ſame number of equal and ſimilar planes, ſimilarly ſituated, ſo that the plane AC is ſimilar and equal to the plane KM, the plane AF to the plane KP; BG to LQ, GD to QN, DE to NO, and FH to PR. Let alſo the inclination of the plane AF to the plane AC be the ſame with that of the plane KP to the plane KM, and ſo of the reſt; the ſolid KQ is equal and ſimilar to the ſolid AG.



Let the ſolid KQ be applied to the ſolid AG, ſo that the baſes KM and AC, which are equal and ſimilar, may coincide[a], the point N coinciding with the point D, K with A, L with B, and ſo on. And becauſe the plane KM coincides with the plane AC, and, by hypotheſis, the inclination of KR to KM is the ſame with the inclination of AH to AC, the plane KR will be upon the plane AH, and will coincide with it, becauſe they are ſimilar and equal[a], and becauſe their equal ſides KN and AD coincide. And in the ſame manner, it is ſhewn that the other planes of the ſolid KQ coincide with the other planes of the ſolid AG, each with each: wherefore the ſolids KQ and AG do wholly coincide, and are equal and ſimilar to one another. Therefore, &c. Q. E. D.

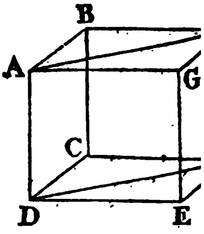
[a] 8. Ax. 1.

PROP. XXII. THEOR.

IF a ſolid be contained by ſix planes, two and
of which are parallel, the oppoſite planes ar
milar and equal parallelograms.

Let the ſolid CDGH be contained by the parallel
AC, GF; BG, CE; FB, AE: its oppoſite planes are
lar and equal parallelograms.

Becauſe the two parallel planes BG, CE, are cut b
a 14. 7. plane AC, their common ſections AB, CD, are parallel
gain, becauſe the two parallel planes BF, AE, are cut
plane AC, their common ſections AD, BC are para
and AB is parallel to CD; therefore AC is a parallelo
In like manner, it may be proved
that each of the figures CE, FG,
GB, BF, AE is a parallelogram:
Join AH, DF; and becauſe AB
is parallel to DC, and BH to CF;
the two ſtraight lines AB, BH,
which meet one another, are parallel
to DC and CF, which
meet one another; wherefore, tho'
the firſt two are not in the ſame plane
with the other two, they contain equal
b 9. 7. angles b; the angle ABH is
therefore equal to the angle DCF.
And becauſe AB, BH, are equal to DC, CF, and the
ABH equal to the angle DCF; therefore the baſe A
c 4. 1. equal c to the baſe DF, and the triangle ABH to th
angle DCF: For the ſame reaſon, the triangle AGH is
to the triangle DEF; and therefore the parallelogram
equal and ſimilar to the parallelogram CE. In the
manner it may be proved, that the parallelogram AC is
and ſimilar to the parallelogram GF, and the parallel
AE to BF. Therefore, if a ſolid, &c. Q. E. D.



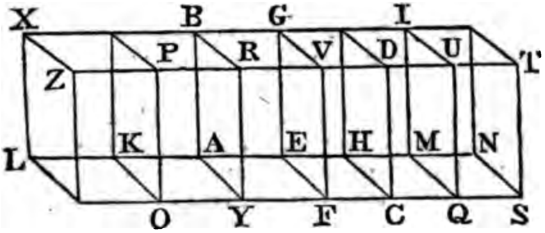
PR

PROP. XXIII. THEOR.

F a ſolid parallelepiped be cut by a plane parallel to two of its oppoſite planes, it will be divided to two ſolids, which will be to one another as eir baſes.

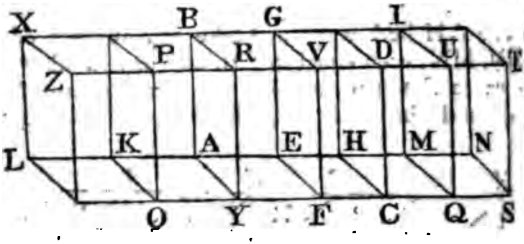
Let the ſolid parallelepiped ABCD be cut by the plane V, which is parallel to the oppoſite planes AR, HD, and ides the whole into the ſolids ABFV, EGCD; as the ſe AEFY to the baſe EHCF, ſo is the ſolid ABFV to the id EGCD.

Produce AH both ways, and take any number of ſtraight es HM, MN, each equal to EH, and any number AK, KL ch equal to EA, and complete the parallelograms LO, KY, Q, MS, and the ſolids LP, KR, HU, MT: then, becauſe



e ſtraight lines LK, KA, AE are all equal, and alſo the ight lines KO, AY, EF, which make equal angles with K, KA, AE, the parallelograms LO, KY, AF are eal and ſimilar [a]: and likewiſe the parallelograms KX, a 20 6.
B, AG; as alſo [b] the parallelograms LZ, KP, AR, b 22. 7.
cauſe they are oppoſite planes. For the ſame reaſon, parallelograms EC, HQ, MS, are equal [a]; and the palelograms HG, HI, IN, as alſo [b] HD, MU, NT; theree three planes of the ſolid LP, are equal and ſimilar three planes of the ſolid KR, as alſo to three planes of ſolid AV: but the three planes oppoſite to theſe three equal and ſimilar to them [b] in the ſeveral ſolids; theree the ſolids LP, KR, AV are contained by equal and ſilar planes. And becauſe the planes LZ, KP, AR are palel, and are cut by the plane XV, the inclination of LZ

to XP is equal to that of KP to PB, or of AR to BV [c]:
the ſame is true of the other contiguous planes, therefore
c 15. 7 ſolids LP, KR, and AV, are equal to one another [d]. For
d 21. 7. ſame reaſon, the three ſolids ED, HU, MT are equal to
another; therefore what multiple ſoever the baſe LF
the baſe AF, the ſame multiple is the ſolid LV of the
AV; for the ſame reaſon, whatever multiple the baſe NF
of the baſe HF, the ſame multiple is the ſolid NV
ſolid ED: and if the baſe LF be equal to the baſe NF,
ſolid LV is equal [d] to the ſolid NV; and if the baſe LF
greater than the baſe NF, the ſolid LV is greater
the ſolid NV; and if leſs, leſs. Since then there are



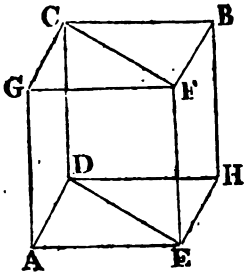
magnitudes, viz. the two baſes AF, FH, and the two
lids AV, ED, and of the baſe AF and ſolid AV, the
LF and ſolid LV are any equimultiples whatever; and
the baſe FH and ſolid ED, the baſe FN and ſolid NV
any equimultiples whatever; and it has been proved, that
the baſe LF is greater than the baſe FN, the ſolid LV
greater than the ſolid NV; and if equal, equal; and if
e 5. def. 5. leſs: Therefore [e], as the baſe AF is to the baſe FH, ſo
the ſolid AV to the ſolid ED. Wherefore, if a ſolid,
Q. E. D.

Cor. Becauſe the parallelogram AF is to the parallelog
f 1. 6. FH as YF to FC [f], therefore the ſolid AV is to the ſolid
as YF to FC.

PROP. XXIV. THEOR.

IF a ſolid parallelepiped be cut by a plane paſſ
through the diagonals of two of the oppo
planes, it ſhall be cut in two equal priſms.

AB be a ſolid parallelepiped, and DE, CF the diago- the oppoſite parallelograms AH, GB, viz. thoſe which wn betwixt the equal angles in each: and becauſe E are each of them parallel to GA, though not in the lane with it, CD, FE are parallel [a]; wherefore the ls CF, DE are in the plane h the parallels are, and are lves parallels [b]: and the plane ſhall cut the ſolid AB into ual parts.



uſe the triangle CGF is equal triangle CBF, and the tri- AE to DHE; and that the logram CA is equal [d] and ſi- o the oppoſite one BE; and rallelogram GE to CH: re the planes which contain ſms CAE, CBE, are equal and ſimilar, each to each; and e alſo equally inclined to one another, becauſe the planes B are parallel, as alſo AF and BD, and they are cut plane CE. Therefore the priſm CAE is equal to the BE [e], and the ſolid AB is cut into two equal priſms plane CDEF. Q. E. D.

a 8. 7.

b 14. 7.

c 34. 1.

d 22. 7.

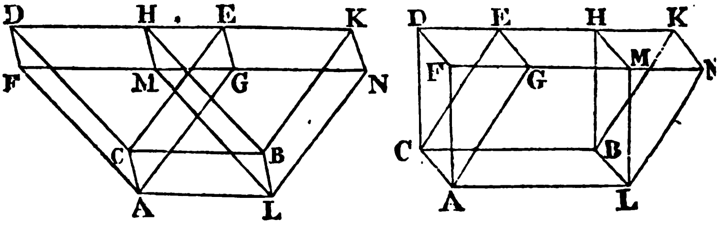
e 21. 7.

The inſiſting ſtraight lines of a parallelepiped, men- in the following propoſitions, are the ſides of the paral- ms betwixt the baſe and the plane parallel to it.

PROP. XXV. THEOR.

SOLID parallelepipeds upon the ſame baſe, an of the ſame altitude, the inſiſting ſtraight lin of which are terminated in the ſame ſtraight lin in the plane oppoſite to the baſe, are equal to o another.

Let the ſolid parallelopipeds AH, AK be upon the ſ
baſe AB, and of the ſame altitude, and let their inſi
ſtraight lines AF, AG, LM, LN, be terminated in the ſa
ſtraight line FN, and CD, CE, BH, BK be terminated in t
ſame ſtraight line DK; the ſolid AH is equal to the ſo
AK.
a 34. 1. Becauſe CH, CK are parallelograms, CB is equal a to ea
of the oppoſite ſides DH, EK; wherefore DH is equal
EK: add, or take away the common part HE; then DE
b 38. 1. equal to HK: Wherefore alſo the triangle CDE is equal b
c 36. 1. the triangle BHK: and the parallelogram DG is equal c
the parallelogram HN. For the ſame reaſon, the triang
AFG is equal to the triangle LMN, and the parallelogr
d 22. 7. CF is equal d to the parallelogram BM, and CG to BN; f



they are oppoſite. Therefore the planes which contain t
priſm DAG are ſimilar and equal to thoſe which contain t
priſm HLN, each to each; and the contiguous planes are al
e 15 7. equally inclined to one another e, becauſe that the parall
planes AD and LH, as alſo AE and LK, are cut by the ſam
f 21. 7. plane DN: therefore the priſms DAG, HLN are equal f.

therefo

therefore the priſm LNH be taken from the ſolid, of which the baſe is the parallelogram AB, and FDKN the plane oppoſite to the baſe; and if from this ſame ſolid there be taken the priſm AGD, the remaining ſolid, viz. the parallelepiped AH, is equal to the remaining parallelepiped AK. Therefore ſolid parallelepipeds, &c. Q. E. D.

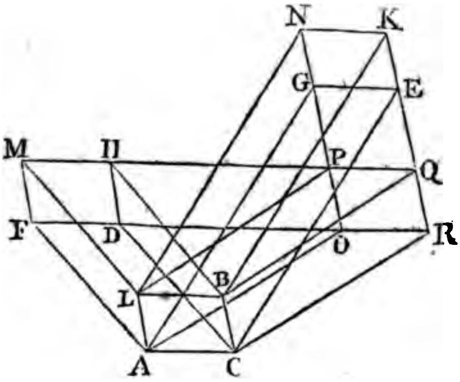
PROP. XXVI. THEOR.

SOLID parallelepipeds upon the ſame baſe, and of the ſame altitude, the inſiſting ſtraight lines of which are not terminated in the ſame ſtraight lines in the plane oppoſite to the baſe, are equal to one another.

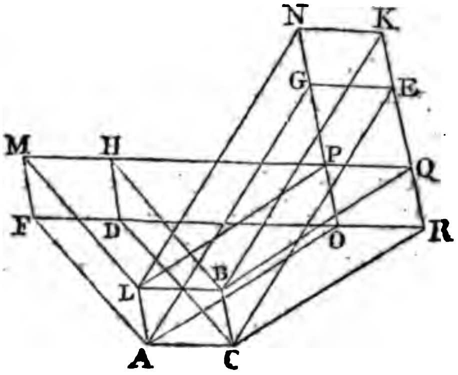
Let the parallelepipeds CM, CN, be upon the ſame baſe AB, and of the ſame altitude, but their inſiſting ſtraight lines AF, AG, LM, LN, CD, CE, BH, BK, not terminated in the ſame ſtraight lines; the ſolids CM, CN are equal to one another.

Produce FD, MH, and NG, KE, and let them meet one another in the points O, P, Q, R; and join AO, LP, BQ, CR. Becauſe the planes[a] LBHM and ACDF are parallel, and becauſe the plane LBHM is that in which are the parallels LB, MHPQ[a], and in which alſo is the figure

a 12. def. 7.



BLPQ; and becauſe the plane ACDF is that in which are the parallels AC, FDOR, and in which alſo is the figure CAOR; therefore the figures BLPQ, CAOR are in parallel planes. In like manner, becauſe the planes ALNG and CBKE are parallel, and the plane ALNG is that in which are the parallels AL, OPGN, and in which alſo is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, and in which alſo is the figure CBQR; therefore the figures ALPO, CBQR are in parallel planes. But the planes ACBL, ORQP are alſo parallel; therefore the ſolid CP is a parallelepiped. Now the ſolid paral-



b 25. 7. lelepiped CM is equal [b] to the ſolid parallelepiped CP; becauſe they are upon the ſame baſe, and their inſiſting ſtraight lines AF, AO, CD, CR; LM, LP, BH, BQ are in the ſame ſtraight lines FR, MQ: and the ſolid CP is equal [b] to the ſolid CN; for they are upon the ſame baſe ACBL, and their inſiſting ſtraight lines AO, AG, LP, LN; CR, GE, BQ, BK are in the ſame ſtraight lines ON, RK: therefore the ſolid CM is equal to the ſolid CN. Wherefore ſolid parallelepipeds, &c. Q. E. D.

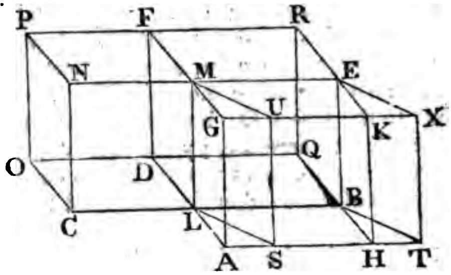
PROP. XXVII. THEOR.

SOLID parallelepipeds which are upon equal baſes, and of the ſame altitude, are equal to one another.

Let the ſolid parallelepipeds, AE, CF, be upon equal baſes AB, CD, and be of the ſame altitude: the ſolid AE is equal to the ſolid CF.

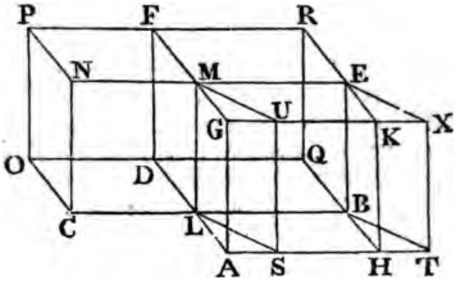
Caſe 1. Let the inſiſting ſtraight lines be at right angles to the baſes AB, CD, and let the baſes be placed in the ſame plane, and ſo as that the ſides CL, LB be in a ſtraight line; therefore the ſtraight line LM, which is at right angles to the plane in which the baſes are, in the point L, is common[a] to the two ſolids AE, CF; let the other inſiſting lines of the ſolids be AG, HK, BE; DF, OP, CN: and firſt, let the angle ALB be equal to the angle CLD; then AL, LD are in a ſtraight line[b]. Produce OD, HB, and let them meet in Q, and complete the ſolid parallelepiped LR, the baſe of which is the parallelogram LQ, and of which LM is one of its inſiſting ſtraight lines: therefore, becauſe the parallelogram AB is equal to CD, as the baſe AB is to the baſe LQ, ſo is[c] the baſe CD to the ſame LQ: and becauſe the ſolid parallelepiped AR is cut by the plane LMEB, which is parallel to the oppoſite planes AK, DR; as the baſe AB is to the baſe LQ, ſo is[d] the ſolid AE to the ſolid LR: for the ſame reaſon, becauſe the ſolid parallelepiped CR is cut by the plane LMFD, which is parallel to the oppoſite planes CP, BR; as the baſe CD to the baſe LQ; ſo is the ſolid CF to the

[a] 11. 7.
[b] 14. 1.
[c] 7. 5.
[d] 13. 7.



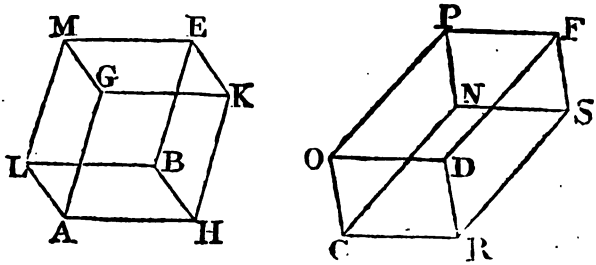
the ſolid LR: but as the baſe AB to the baſe LQ, ſo the baſe CD to the baſe LQ, as has been proved: therefore as the ſolid AE to the ſolid LR, ſo is the ſolid CF to the ſolid e 9. 5. LR; and therefore the ſolid AE is equal[e] to the ſolid CF.

But let the ſolid parallelepipeds SE, CF be upon equal baſes SB, CD, and be of the ſame altitude, and let their inſiſting ſtraight lines be at right angles to the baſes; and place he baſes SB, CD in the ſame plane, ſo that CL, LB be in a ſtraight line; and let the angles SLB, CLD be unequal; the ſolid SE is alſo in this caſe equal to the ſolid CF. Produce DL, TS until they meet in A, and from B draw BH parallel to DA; and let HB, OD produced meet in Q, and complete the ſolids AE, LR: therefore the ſolid AE, of which the baſe is the parallelogram LE, and AK the plane oppoſite f 25. 7. to it, is equal[f] to the ſolid SE, of which the baſe is LE, and SX the plane oppoſite; for they are upon the ſame baſe LE, and of the ſame altitude, and their inſiſting ſtraight lines, viz. LA, LS, BH, BT; MG, MU, EK, EX are in the ſame ſtraight lines AT, GX: and becauſe the parallelogram AB is equal[g] g 35. 1. to SB, for they are upon the ſame baſe LB, and between the ſame parallels LB, AT; and becauſe the baſe SB is equal to the baſe CD; therefore the baſe AB is equal to the baſe CD; but the angle ALB is equal to the angle CLD: therefore, by the firſt caſe, the ſolid AE is equal to the ſolid CF; but the ſolid AE is equal to the ſolid SE, as was demonſtrated; therefore the ſolid SE is equal to the ſolid CF.



Caſe 2. If the inſiſting ſtraight lines AG, HK, BE, LM; CN, RS, DF, OP, be not at right angles to the baſes AB, CD; in this caſe likewiſe the ſolid AE is equal to the ſolid CF. Becuſe ſolid parallelepipeds on the h 26. 7 ſame baſe, and of the ſame altitude, are equal[h], if two ſolid

lid parallelepipeds be conſtituted on the baſes AB and CD



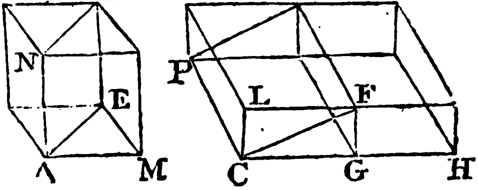
of the ſame altitude with the ſolids AE and CF, and with their inſiſting lines perpendicular to their baſes, they will be equal to the ſolids AE and CF; and, by the firſt caſe of this propoſition, they will be equal to one another; wherefore, the ſolids AE and CF are alſo equal. Wherefore, ſolid parallelepipeds, &c. Q. E. D.

PROP. XXVIII. THEOR.

SOLID parallelepipeds which have the ſame altitude, are to one another as their baſes.

Let AB, CD be ſolid parallelepipeds of the ſame altitude: they are to one another as their baſes; that is, as the baſe AE to the baſe CF, ſo is the ſolid AB to the ſolid CD.

To the ſtraight line FG apply the parallelogram FH equal [a] to AE, ſo that the angle FGH be equal to the angle LCG; and complete the ſolid parallelepiped GK upon the baſe FH, one of whoſe inſiſting lines is FD, whereby the ſolids CD, GK muſt be of the ſame altitude. Therefore the ſolid



[a] cor. 45. 1.

ſolid AB is equal [b] to the ſolid GK, becauſe they are upon equal baſes AE, FH, and are of the ſame altitude: and becauſe the ſolid parallelepiped CK is cut by the plane DG which is parallel to its oppoſite planes, the baſe HF is [c] to the baſe FC, as the ſolid HD to the ſolid DC: But the baſe HF is equal to the baſe AE, and the ſolid GK to the ſolid AB: therefore, as the baſe AE to the baſe CF, ſo is the ſolid AB to the ſolid CD. Wherefore ſolid parallelepipeds, &c. Q. E. D.

b 27. 7.
c 23. 7.

Cor. 1. From this it is manifeſt, that priſms upon triangular baſes, of the ſame altitude, are to one another as their baſes. Let the priſms BNM, DPG the baſes of which are the triangles AEM, CFG, have the ſame altitude; complete the parallelograms AE, CF, and the ſolid parallelepipeds AB, CD, in the firſt of which let AN, and in the other let CP be one of the inſiſting lines. And becauſe the ſolid parallelepipeds AB, CD have the ſame altitude, they are to one another as the baſe AE is to the baſe CF; wherefore the priſms, which are their halves [d] are to one another, as the baſe AE to the baſe CF; that is, as the triangle AEM to the triangle CFG.

d 24. 7.

Cor. 2. Alſo a priſm and a parallelepiped, which have the ſame altitude, are to one another as their baſes; that is, the priſm BNM is to the parallelepiped CD as the triangle AEM to the parallelogram LG. For by the laſt Cor. the priſm BNM is to the priſm DPG as the triangle AME to the triangle CGF, and therefore the priſm BNM is to twice the priſm DPG as the triangle AME to twice the triangle CGF [e]; that is, the priſm BNM is to the parallelepiped CD as the triangle AME to the parallelogram LG.

e 4. 5.

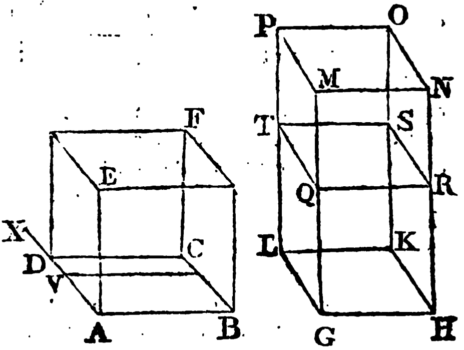
PROP. XXIX. THEOR.

SOLID parallelepipeds are to one another in the ratio that is compounded of the ratios of their baſes, and of their altitudes.

Let AF and GO be two ſolid parallelepipeds, of which the baſes are the parallelograms AC and GK, and the altitudes, the perpendiculars let fall on the planes of theſe baſes from any point in the oppoſite planes EF and MO; the ſolid AF is to the ſolid GO in a ratio compounded of the ratios of the baſe AC to the baſe GK, and of the perpendicular on AC to the perpendicular on GK.

Caſe 1. When the inſiſting lines are perpendicular to the baſes AC and GK, or when the ſolids are upright.

In GM, one of the inſiſting lines of the ſolid GO, take GQ equal to AE, one of the inſiſting lines of the ſolid AF, and through Q let a plane paſs parallel to the plane GK, meeting the other inſiſting lines of the ſolid GO in the points R, S and T. It is evident that GS is a ſolid parallelepiped[a], and that it has the ſame altitude with AF, viz. GQ or AE. Now the ſolid AF is to the ſolid GO in a ratio compounded of the ratios of the ſolid AF to the ſolid GS[b], and of the ſolid GS to the ſolid GO; but the ratio of the ſolid AF to the ſolid GS, is the ſame with that of the baſe of AC to the baſe GK[c], becauſe their altitudes AE and GQ are equal; and the ratio of the ſolid GS to the ſolid GO, is the ſame with that of GQ to GM[d]; therefore, the ratio which is compounded of the ratios of the ſolid AF to the ſolid GS, and of the ſolid GS to

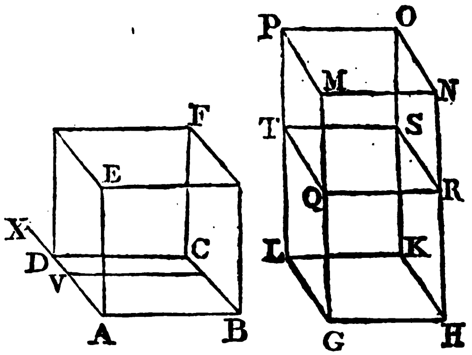


a 12. 7.
b 10. def. 5.
c 28. 7.
d 23. 7.

the ſolid GO, is the ſame with the ratio which is compounded of the ratios of the baſe AC to the baſe GK,
e E. 5. and of the altitude AE to the altitude GM [e]. But the ratio of the ſolid AF to the ſolid GO, is that which is compounded of the ratios of AF to GS, and of GS to GO; therefore, the ratio of the ſolid AF to the ſolid GO is compounded of the ratios of the baſe of AC to the baſe GK, and of the altitude AE to the altitude GM.

Caſe 2. When the inſiſting lines are not perpendicular to the baſes.

Let the parallelograms AC and GK be the baſes as before, and let AE and GM be the altitudes of two parallelepipeds Y and Z on theſe baſes. Then, if the upright parallelepipeds AF and GO be conſtituted on the baſes AC and GK, with the altitudes AE and GM, they will be



d 27. 7. equal to the parallelepipeds Y and Z [d]. Now, the ſolids AF and GO, by the firſt caſe, are in the ratio compounded of the ratios of the baſes AC and GK, and of the altitudes AE and GM; therefore alſo, the ſolids Y and Z have to one another a ratio that is compounded of theſe ſame ratios. Therefore, &c. Q. E. D.

Cor. 1. Hence, two ſtraight lines may be found having the ſame ratio with the two parallelepipeds AF and GO. To AB, one of the ſides of the parallelogram AC, apply the parallelogram BV equal to GK, having an angle equal to
e 44. 1. the angle BAD [e]; and as AE to GM, ſo let AV be to AX [f],
f 12. 6. then AD is to AX as the ſolid AF to the ſolid GO. For
g 10. def. 5. the ratio of AD to AX is compounded of the ratios [g] of AD to AV, and of AV to AX; but the ratio of AD to AV is the ſame with that of the parallelogram AC to the parallelo-
h 1. 6. gram BV [h] or GK; and the ratio of AV to AX is the ſame with that of AE to GM; therefore, the ratio of AD to AX
k E. 5. is compounded of the ratios of AC to GK, and of AE to GM [k].
But

But the ratio of the ſolid AF to the ſolid GO is compounded of the ſame ratios; therefore, as AD to AX, ſo is the ſolid AF to the ſolid GO.

Cor. 2. Hence all priſms are to one another in the ratio compounded of the ratios of their baſes, and of their altitudes. For every priſm is equal to a parallelepiped of the ſame altitude with it, and of an equal baſe[l]. l 28. cor. 2. 7.

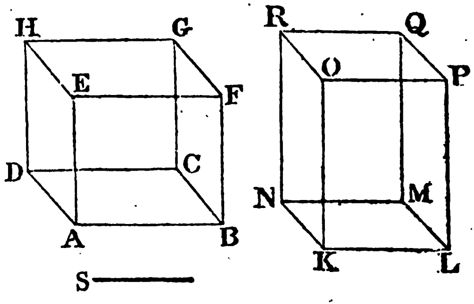
PROP. XXX. THEOR.

SOLID parallelepipeds which have their baſes and altitudes reciprocally proportional, are equal; and parallelepipeds which are equal, have their baſes and altitudes reciprocally proportional.

Let AG and KQ be two ſolid parallelepipeds, of which the baſes are AC and KM, and the altitudes AE and KO, and let AC be to KM as KO to AE, the ſolids AG and KQ are equal.

As the baſe AC to the baſe KM, ſo let the ſtraight line KO be to the ſtraight line S. Then, ſince AC is to KM as KO to S, and alſo by hypotheſis, AC to KM as KO to AE, KO has the ſame ratio to S that it has to AE[a]; wherefore AE is equal to S[b]. But the ſolid AG is to the ſolid KQ in the ratio compounded of the ratios of AE to KO, and of AC to KM[c], that is, in the ratio compounded of the ratios of AE to KO, and of KO to S. And the ratio of AE to S is alſo compounded of the ſame ratios[d]; therefore, the ſolid AG has to the ſolid KQ the ſame ratio that AE has to S. But AE was proved to be equal to S, therefore AG is equal to KQ.

a 11. 5.
b 9. 5.
c 29. 7.
d 10. def. 5.



Again, if the ſolids AG and KQ be equal, the baſe AC is to the baſe KM as the altitude KO to the altitude AE. Take S, ſo that AC may be to KM as KO to S, and it will be ſhewn, as was done above, that the ſolid AG is to the ſolid KQ as AE to S; now, the ſolid AG is, by hypotheſis, equal to the ſolid KQ; therefore, AE is equal to S; but, by conſtruction, AC is to KM as KO is to S; therefore, AC is to KM as KO to AE. Therefore, &c. Q. E. D.

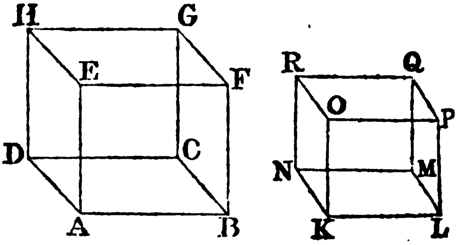
Cor. In the ſame manner, may it be demonſtrated, that equal priſms have their baſes and altitudes reciprocally proportional, and converſely.

PROP. XXXI. THEOR.

SIMILAR ſolid parallelepipeds are to one another in the triplicate ratio of their homologous ſides.

Let AG, KQ be two ſimilar parallelepipeds, of which AB and KL are two homologous ſides; the ratio of the ſolid AG to the ſolid KQ is triplicate of the ratio of AB to KL.

a 9. def. 7. b 1. def. 6. Becauſe the ſolids are ſimilar, the parallelograms AF, KP are ſimilar [a], as alſo the parallelograms AH, KR; therefore, the ratios of AB to KL, of AE to KO, and of AD to KN, are all equal. But the ratio of the ſolid AG to the ſolid KQ is compounded of the ratios of AC to KM, and of AE to KO. Now, the ratio of AC to KM, becauſe they are



c 14. 6. equiangular parallelograms, is compounded [c] of the ratios of AB to KL, and of AD to KN. Wherefore, the ratio of AG to KQ is compounded of the three ratios of AB to KL, AD to KN, and AE to KO; and theſe three ratios have already been proved to be equal; therefore, the ratio that is compounded of them, viz. the ratio of the ſolid AG to the ſolid KQ, is triplicate of any of them d; it is therefore triplicate of the ratio of AB to KL. Therefore, ſimilar ſolid parallelepipeds, &c. Q. E. D.

d 12 def. 5.

Cor. 1. If as AB to KL, ſo KL to *m*, and as KL to *m*, ſo is *m* to *n*, then AB is to *n* as the ſolid AG to the ſolid KQ. For the ratio of AB to *n* is triplicate of the ratio of AB to KL e, and is therefore equal to that of the ſolid AG to the ſolid KQ.

e cor. 12. def. 5.

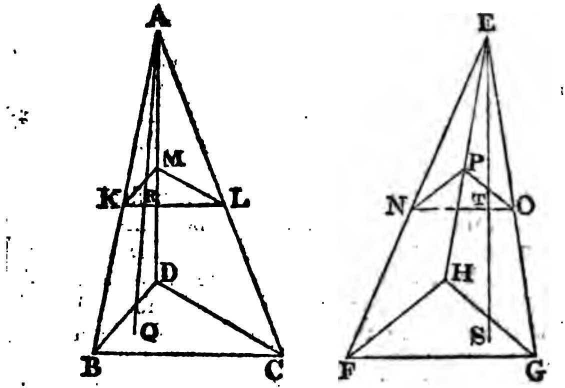
Cor. 2. As cubes are ſimilar ſolids, therefore the cube on AB is to the cube on KL in the triplicate ratio of AB to KL, that is in the ſame ratio with the ſolid AG to the ſolid KQ. Similar ſolid papallelepipeds are therefore to one another as the cubes on their homologous ſides.

Cor. 3. In the ſame manner it is proved, that ſimilar priſms are to one another in the triplicate ratio, or in the ratio of the cubes, of their homologous ſides.

PROP. XXXII. THEOR.

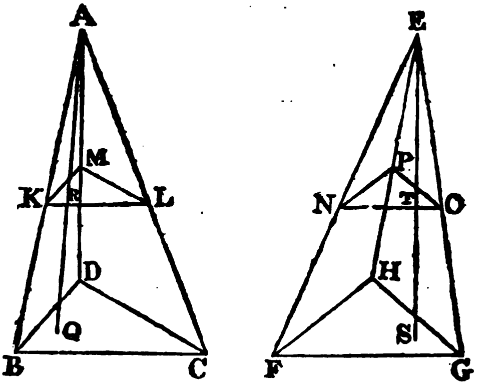
IF two triangular pyramids which have equal baſes and altitudes be cut by planes that are parallel to the baſes, and at equal diſtances from them, the ſections are equal to one another.

Let ABCD and EFGH be two pyramids, having equal baſes BDC and FGH, and equal altitudes, viz. the perpendiculars AQ and ES drawn from A and E upon the planes



BDC and FGH ; and let them be cut by planes parallel to BDC and FGH, and at equal altitudes QR and ST above thoſe planes, and let the ſections be the triangles KLM, NOP ; KLM and NOP are equal to one another.

Becauſe the plane ABD cuts the parallel planes BDC,
a 14. 7. KLM, the common ſections BD and KM are parallel [a]. For the ſame reaſon, DC and ML are parallel. Since therefore KM and ML are parallel to BD and DC, each to each, though not in the ſame plane with them, the angle KML is equal to the
b 9. 7 angle BDC [b]. In like manner the other angles of theſe triangles are proved to be equal ; therefore, the triangles are equiangular, and conſequently ſimilar ; and the ſame is true of the triangles NOP, FGH.



Now, ſince the ſtraight lines ARQ, AKB meet the parallel planes BDC and KML, they are cut by them propor-
c 16. 7. tionally [c], or QR : RA : : BK : KA ; and AQ : AR : : AB
d 18. 5. : AK [d], for the ſame reaſon, ES : ET : : EF : EN ; therefore, AB : AK : : EF : EN, becauſe AQ is equal to ES, and AR to ET. Again, becauſe the triangles ABC, AKL are ſimilar

AB : AK : : BC : KL ; and for the ſame reaſon
EF : EN : : FG : NO ; therefore,
BC : KL : : FG : NO. And, when four ſtraight lines are proportionals, the ſimilar figures deſcribed on them are
e 22. 6. alſo proportionals [e] ; therefore, the triangle BCD is to the triangle KLM as the triangle FGH to the triangle NOP ; but the triangles BDC, FGH are equal ; therefore, the tri-
f 14. 5 angle KLM is alſo equal to the triangle NOP [f]. Therefore, &c. Q. E. D.

COR. 1. Becauſe it has been ſhewn that the triangle KLM is ſimilar to the baſe BCD, therefore, any ſection of a triangular pyramid parallel to the baſe, is a triangle ſimilar to the baſe. And in the ſame manner it is ſhewn, that the ſections parallel to the baſe of a polygonal pyramid are ſimilar to the baſe.

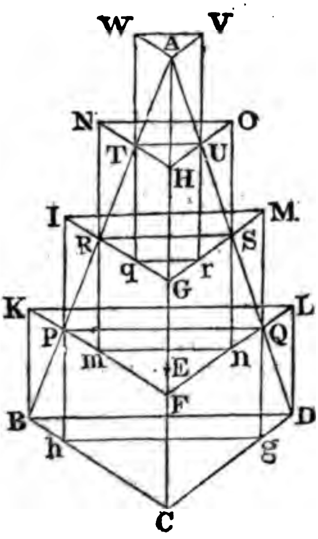
Cor. 2. Hence alſo, the ſections parallel to the baſes of two polygonal pyramids, and at equal diſtances from the baſes, are equal to one another.

PROP. XXXIII. THEOR.

A Series of priſms of the ſame altitude may be circumſcribed about any pyramid, ſuch that the ſum of the priſms ſhall exceed the pyramid by a ſolid leſs than any given ſolid.

Let ABCD be a pyramid, and Z † a given ſolid; a ſeries of priſms having all the ſame altitude, may be circumſcribed about the pyramid ABCD, ſo that their ſum ſhall exceed ABCD by a ſolid leſs than Z.

Let Z be equal to a priſm ſtanding on the ſame baſe with the pyramid, viz. the triangle BCD, and having for its altitude the perpendicular drawn from a certain point E in the line AC upon the plane BCD. It is evident, that CE multiplied by a certain number *m* will be greater than AC; divide CA into as many equal parts as there are units in *m*, and let theſe be CF, FG, GH, HA, each of which will be leſs than CE. Through each of the points F, G, H let planes be made to paſs parallel to the plane BCD making with the ſides of the pyramid the ſections FPQ, GRS, HTU, which will be all ſimilar to one another, and to the baſe BCD [a]. From the point B draw in the plane of the triangle ABC the ſtraight line BK parallel to



[a] 1. cor. 32. 7.

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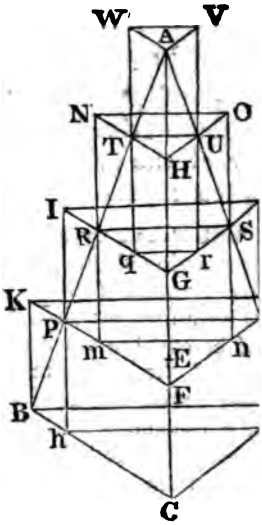
† The ſolid Z is not repreſented in the figure of this or the following Propoſition.

Book VII. CF meeting FP produced in K. In like manner, fro
draw DL parallel to CF, meeting FQ in L: Join KL,
b 11. def. 7. it is plain, that the ſolid KBCDLF is a priſm [b]. By the
conſtruction, let the priſms PM, RO, TV be deſcribed.
ſo, let the ſtraight line IP, which is in the plane of the triangle ABC be produced till it meet BC in h; and let MQ be produced till it meet DC in g: Join hg; then hCgQFP is a priſm, and is equal to the
d 1. cor. 28 7. priſm PM [d]. In the ſame manner is deſcribed the priſm mS equal to the priſm RO, and the priſm qU equal to the priſm TV. The ſum, therefore, of all the inſcribed priſms hQ, mS, and qU is equal to the ſum of the priſms PM, RO and TV, that is, to the ſum of all the circumſcribed priſms except the priſm BL; wherefore, BL is the exceſs of the priſms circumſcribed about the pyramid ABCD above the priſms inſcribed within it. But the priſm BL is leſ
the priſm which has the triangle BCD for its baſe, and
altitude the perpendicular from E upon the plane BCD
the priſm which has BCD for its baſe, and the perpenc
from E for its altitude, is by hypotheſis equal to the give
Z; therefore, the exceſs of the circumſcribed, above t
ſcribed priſms is leſs than the given ſolid Z. But the
of the circumſcribed priſms above the inſcribed is greate
their exceſs above the pyramid ABCD, becauſe AB
greater than the ſum of the inſcribed priſms. Much
therefore, is the exceſs of the circumſcribed priſms abo
pyramid, leſs than the ſolid Z. A ſeries of priſms of th
altitude has therefore been circumſcribed about the py
ABCD exceeding it by a ſolid leſs than the given ſo
Q. E. D.



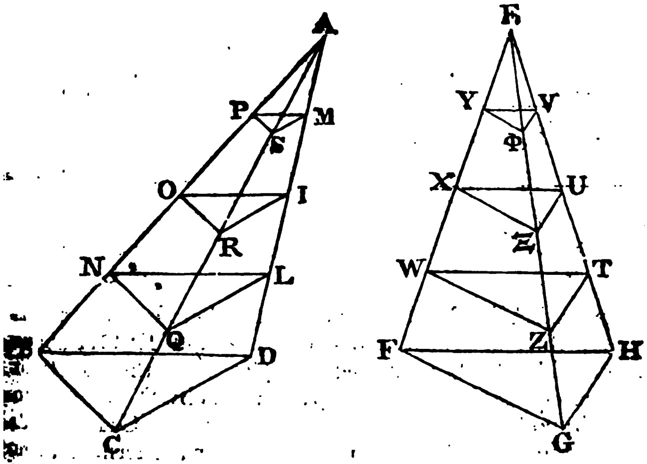
P I

PROP. XXXIV. THEOR.

PYRAMIDS that have equal baſes and altitudes are equal to one another.

Let ABCD, EFGH be two pyramids that have equal baſes BCD, FGH, and alſo equal altitudes, viz. the perpendiculars drawn from the vertices A and E upon thc planes BCD, FGH: The pyramid ABCD is equal to the pyramid EFGH.

If they are not equal, let the pyramid EFGH exceed the pyramid ABCD by the ſolid Z. Then, a ſeries of priſms of the ſame altitude may be deſcribed about the pyramid ABCD that ſhall exceed it, by a ſolid leſs than Z[a]; let theſe be the priſms that have for their baſes the triangles BCD, NQL, ORI, PSM. Divide EH into the ſame number of equal parts into which AD is divided, viz. HT, TU, UV, VE, and through the points T, U and V, let the ſections TZW, UΞX, VΦY be made parallel to the baſe FGH. The ſection NQL is equal to the ſection WZT[b]; as alſo ORI to XΞU, and PSM to YΦV; and therefore, alſo the priſms that ſtand upon the equal ſections are



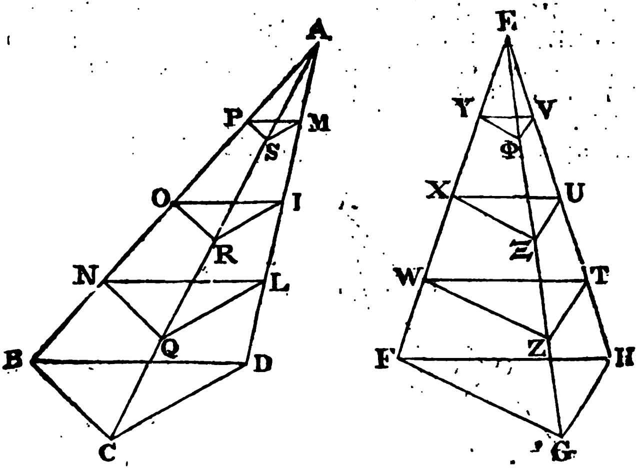
[a] 33. 7.

[b] 32. 7.

Book VII.

c 1. cor. 28. 7.

are equal c, that is, the priſm which ſtands on the baſe BCD, and which is between the planes BCD and NQL is equal to the priſm which ſtands on the baſe FGH, and which is between the planes FGH and WZT; and ſo of the reſt, becauſe they have the ſame altitude; wherefore, the ſum of all the priſms deſcribed about the pyramid ABCD is equal to the ſum of all thoſe deſcribed about the pyramid EFGH. But the exceſs of the priſms deſcribed about the pyramid ABCD above



the pyramid ABCD is leſs than Z; and therefore, the exceſs of the priſms deſcribed about the pyramid EFGH above the pyramid ABCD is alſo leſs than Z. But the exceſs of the pyramid EFGH above the pyramid ABCD is equal to Z, by hypotheſis; therefore, the pyramid EFGH exceeds the pyramid ABCD, more than the priſms deſcribed about EFGH exceed the ſame pyramid ABCD: The pyramid EFGH is therefore greater than the ſum of the priſms deſcribed about it, which is impoſſible. The pyramids ABCD, EFGH, therefore, are not unequal, that is, they are equal to one another. Therefore, pyramids, &c. Q. E. D.

PROP. XXXV. THEOR.

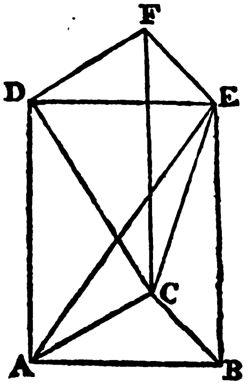
ERY priſm having a triangular baſe may be divided into three pyramids that have triangu- ſes, and that are equal to one another.

there be a priſm of which the baſe is the triangle ABC, DEF be the triangle oppoſite to the baſe : The priſm EF may be divided into three equal pyramids having lar baſes.

AE, EC, CD; and becauſe ABED is a parallelo- of which AE is the diameter, the triangle ADE is to the triangle ABE : therefore the pyramid of which ſe is the triangle ADE, and vertex the point C, is to the pyramid, of which the the triangle ABE, and vertex int C. But the pyramid of the baſe is the triangle ABE, rtex the point C, that is, the id ABCE is equal to the py- DEFC [f], for they have equal viz. the triangles ABC, DFE, e ſame altitude, viz. the alti- f the priſm ABCDEF. There- the three pyramids ADEC, , DFEC are equal to one a- . But the pyramids ADEC, , DFEC make up the whole ABCDEF; therefore, the ABCDEF is divided into three equal pyramids. efore, &c. Q. E. D.

a 34. 1.

b 34. 7.



. 1. From this it is manifeſt, that every pyramid is ird part of a priſm which has the ſame baſe, and the altitude with it; for if the baſe of the priſm be any figure than a triangle, it may be divided into priſms triangular baſes.

. 2. Pyramids of equal altitudes are to one another as aſes; becauſe the priſms upon the ſame baſes, and ſame altitude, are [c] to one another as their baſes.

c 1. cor. 28. 7.

ELEMENTS

OF

GEOMETRY.

BOOK VIII.

DEFINITIONS.

I.

AN arch of a circle is any part of the circumference.

II.

The perimeter of any figure is the length of the line or lines by which it is bounded.

AXIOM.

The leaſt line that can be drawn between two points, is a ſtraight line; and if two figures have the ſame ſtraight line for their baſe, that which is contained within the other, if its bounding line or lines be not any where convex toward the baſe, has the leaſt perimeter.

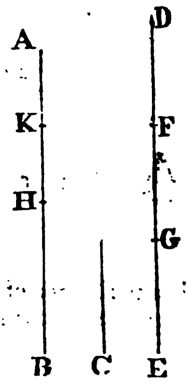
Cor. 1. Hence, the perimeter of any polygon inſcribed in a circle is leſs than the circumference of the circle.

Cor. 2. If from a point two ſtraight lines be drawn touching a circle, theſe two lines are together greater than the arch intercepted between them; and hence, the perimeter of any polygon deſcribed about a circle is greater than the circumference of the circle.

PROP. I. THEOR.

If from the greater of two unequal magnitudes there be taken away its half, and from the remainder its half; and ſo on: There ſhall at length remain a magnitude leſs than the leaſt of the propoſed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater. If from AB there be taken away its half, and from the remainder its half, and ſo on; there ſhall at length remain a magnitude leſs than C.



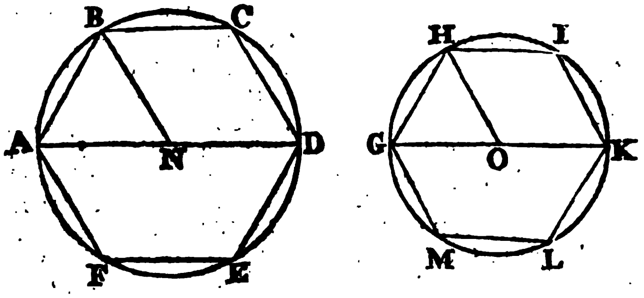
For C may be multiplied ſo as at length to become greater than AB. Let it be ſo multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From AB take BH equal to its half, and from the remainder, AH take HK equal to its half, and ſo on, until there be as many diviſions in AB as there are in DE: And let the diviſions in AB be AK, KH, HB; and the diviſions in ED be DF, FG, GE. And becauſe DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is equal to its half; therefore the remainder GD is greater than the remainder HA. Again, becauſe GD is greater than HA, and that GF is not greater than the half of GD, but HK is equal to the half of HA; therefore, the remainder FD is greater than the remainder AK. And FD is equal to C, therefore C is greater than AK; that is, AK is leſs than C. Q. E. D.

PROP. II. THEOR.

EQUILATERAL polygons of the ſame number of ſides inſcribed in circles, are ſimilar, and are to one another as the ſquares of the diameters of the circles.

Let ABCDEF and GHIKLM be two equilateral polygons of the ſame number of ſides inſcribed in the circles ABD, and GHK; ABCDEF and GHIKLM are ſimilar, and are to one another as the ſquares of the diameters of the circles ABD, GHK.

Find N and O the centres of the circles; join AN and BN, as alſo GO and HO, and produce AN and GO till they meet the circircumferences in D and K.

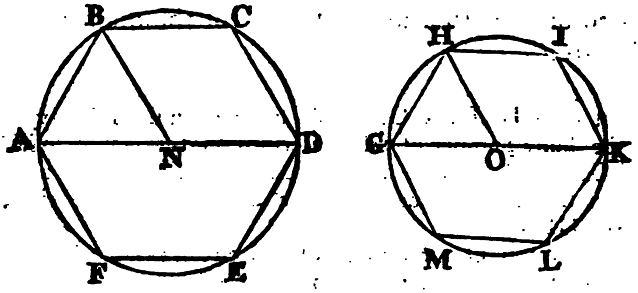


Becauſe the ſtraight lines AB, BC, CD, DE, EF, FA, are all equal, the arches AB, BC, CD, DE, EF, FA are alſo equal [a]. For the ſame reaſon, the arches GH, HI, IK, KL, LM, MG are all equal, and they are equal in number to the others; therefore, whatever part the arch AB is of the whole circumference ABD, the ſame is the arch GH of the circumference GHK. But the angle ANB is the ſame part of four right angles, that the arch AB is of the circumference ABD [b]; and the angle GOH is the ſame part of four right angles that the arch GH is of the circumference GHK [b], therefore the angles ANB, GOH are each of them the ſame part

[a] 28. 3.

[b] 27. 3.

Book VIII. part of four right angles, and therefore they are equal to one another. The iſoſceles triangles ANB, GOH are
c 6. 6. therefore equiangular c, and the angle ABN equal to the angle GHO; in the ſame manner, by joining NC, OE, it may be proved that the angles NBC, OHI are equal



to one another, and to the angle ABN. Therefore the whole angle ABC is equal to the whole GHK; and the ſame may be proved of the angles BCD, HIK, and of the reſt. Therefore, the polygons ABCDEF and GHIKLM are equiangular to one another; and ſince they are equilateral, the ſides about the equal angles are proportionals; the polygon
d 1. def. 6. ABCD is therefore ſimilar to the polygon GHIKLM d. And becauſe ſimilar polygons are as the ſquares of their homolo-
e 20. 6. gous ſides e, the polygon ABCDEF is to the polygon GHIKLM as the ſquare of AB to the ſquare of GH; but becauſe the triangles ANB, GOH are equiangular, the ſquare of AB is to
f 4. 6. the ſquare of GH as the ſquare of AN to the ſquare of GO f,
g 15. 5. or as four times the ſquare of AN to four times the ſquare g of
h 2. cor 8. GO, that is, as the ſquare of AD to the ſquare of GK h. There-
2. fore alſo, the polygon ABCDEF is to the polygon GHIKLM as the ſquare of AD to the ſquare of GK; and they have alſo been ſhewn to be ſimilar. Therefore, &c. Q. E. D.

Cor. Every equilateral polygon inſcribed in a circle is alſo equiangular. For the iſoſceles triangles, which have their common vertex in the centre, are all equal and ſimilar; therefore, the angles at their baſes are all equal, and the angles of the polygon are therefore alſo equal.

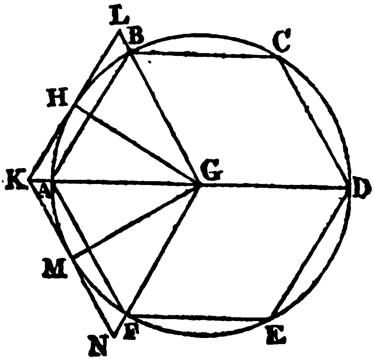
PROP. III. PROB.

THE ſide of any equilateral polygon inſcribed in a circle being given, to find the ſide of a polygon of the ſame number of ſides deſcribed about the circle.

Let ABCDEF be an equilateral polygon inſcribed in the circle ABD; it is required to find the ſide of an equilateral polygon of the ſame number of ſides deſcribed about the circle.

Find G the centre of the circle; join GA, GB, biſect the arch AB in H; and through H draw KHL touching the circle in H, and meeting GA and GB produced in K and L; KL is the ſide of the polygon required.

Produce GF to N, ſo that GN may be equal to GL; join KN, and from G draw GM at right angles to KN, join alſo HG.



Becauſe the circumference AB is biſected in H, the angle AGH is equal to the angle BGH[a]; and becauſe KL touches a 27. 3 the circle in H, the angles LHG, KHG are right angles[b]; therefore, there b 16 3. are two angles of the triangle HGK, equal to two angles of the triangle HGL, each to each. But the ſide GH is common to theſe triangles; therefore they are equal[c], and GL c 26. 1. is equal to GK. Again, in the triangles KGL, KGN, becauſe GN is equal to GL, and GK common, and alſo the angle LGK equal to the angle KGN; therefore the baſe KL is equal to the baſe KN[d]. But becauſe the triangle KGN is iſoſceles, d 4. 1. the angle GKN is equal to the angle GNK, and the angles GMK, GMN are both right angles by conſtruction; wherefore, the triangles GMK, GMN have two angles of the one equal to two angles of the other, and they have

Book VIII.

have alſo the ſide GM common, therefore they are equal[c], and the ſide KM is equal to the ſide MN, ſo that KN is biſected in M. But KN is equal to KL, and therefore their halves KM and KH are alſo equal. Wherefore, in the triangles GKH, GKM, the two ſides GK and KH are equal to the two GK and KM, each to each; and the angles GKH, GKM are alſo equal, therefore GM is equal to GH[d]; wherefore, the point M is in the circumference of the circle; and becauſe KMG is a right angle, KM touches the circle. And in the ſame manner, by joining the centre and the other angular points of the inſcribed polygon, an equilateral polygon may be deſcribed about the circle, the ſides of which will each be equal to KL, and will be equal in number to the ſides of the inſcribed polygon. Therefore, KL is the ſide of an equilateral polygon deſcribed about the circle of the ſame number of ſides with the inſcribed polygon ABCDEF; which was to be found.

c 26. 1.
d 4. 1.

Cor. Becauſe GL, GK, GN, and the other ſtraight lines drawn from the centre G to the angular points of the polygon deſcribed about the circle ABD are all equal; if a circle be deſcribed from the centre G, with the diſtance GK, the polygon will be inſcribed in that circle; and therefore, it is ſimilar to the polygon ABCDEF[e].

e 2. 8.

PROP. IV. THEOR.

A Circle being given, two ſimilar polygons may be found, the one deſcribed about the circle, and the other inſcribed in it, which ſhall differ from one another by a ſpace leſs than any given ſpace.

Let ABC be the given circle, and the ſquare of D any given ſpace; a polygon may be inſcribed in the circle ABC, and a ſimilar polygon deſcribed about it, ſo that the difference between them ſhall be leſs than the ſquare of D.

In the circle ABC apply the ſtraight line AE equal to D, and let AB be a fourth part of the circumference of the circle. From the circumference AB take away its half, and from the remainder its half, and ſo on till the circumference AF is found leſs than the circumference AE[a]. Find the centre G; draw the diameter AC, as alſo the ſtraight lines AF and FG; and having biſected the circumference AF in K, join

a 1. 8.

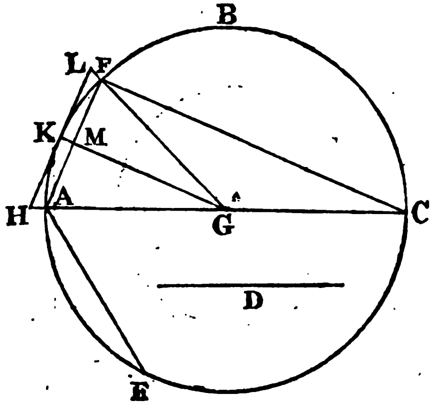
join KG, and draw HL touching the circle in K, and meeting GA and GF produced in H and L; join CF.

Becauſe the iſoſceles triangles HGL and AGF have the common angle AGF, they are equiangular[b], and the angles GHK, GAF are therefore equal to one another. But the angles GKH, CFA are alſo equal, for they are right angles; therefore the triangles HGK, ACF are likewiſe equiangular[c].

[b] 6. 6.

[c] 32. 1.

And becauſe the arch AF was found by taking from the arch AB its half, and from that remainder its half, and ſo on, AF will be contained a certain number of times exactly in the arch AB, and therefore it will alſo be contained a certain number of times exactly in the whole circumference ABC; and the ſtraight line AF is therefore the ſide of an equilateral polygon inſcribed in the circle ABC. Wherefore alſo, HL is the ſide of an equilateral polygon of the ſame number of ſides deſcribed about ABC[d]. Let the polygon deſcribed about the circle be called M, and the polygon inſcribed be called N; then, becauſe theſe polygons are ſimilar[e], they are as the ſquares of the homologous ſides HL and AF[f], that is, becauſe the triangles HLG, AFG are ſimilar, as the ſquare of HG to the ſquare of AG, that is of GK. But the triangles HGK, ACF have been proved to be ſimilar, and therefore, the ſquare of AC is to the ſquare of CF as the polygon M to the polygon N; and, by converſion, the ſquare of AC is to its exceſs above the ſquare of CF, that is, to the ſquare of AF[g], as the polygon M to its exceſs above the polygon N. But the ſquare of AC, that is, the ſquare deſcribed about the circle ABC is greater than the equilateral polygon of eight ſides deſcribed about the circle, becauſe it contains that polygon; and, for the ſame reaſon, the polygon of eight ſides is greater than the polygon of ſixteen, and ſo on; therefore, the ſquare of AC is greater than any



[d] 3. 8.

[e] cor. 3. 8.

[f] 3. cor. 20. 6.

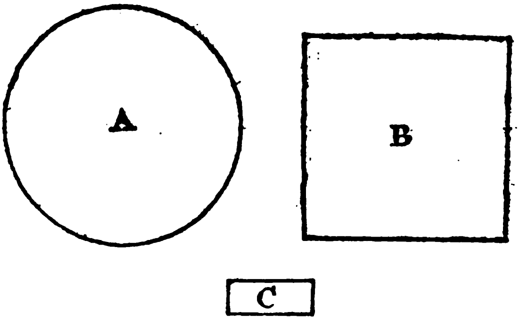
[g] 47. 1.

any polygon deſcribed about the circle by the continual biſection of the arch AB; it is therefore greater than the polygon M. Now, it has been demonſtrated, that the ſquare of AC is to the ſquare of AF as the polygon M to the difference of the polygons; therefore, ſince the ſquare of AC is greater than M, the ſquare of AF is greater than the difference of the polygons [h]. The difference of the polygons is therefore leſs than the ſquare of AF; but AF is leſs than D; therefore, the difference of the polygons is leſs than the ſquare of D, that is, than the given ſpace. Therefore, &c. Q. E. D.

[h] 14. 5.

Cor. 1. Becauſe the polygons M and N differ from one another more than either of them differs from the circle, the difference between each of them and the circle is leſs than the given ſpace, viz. the ſquare of D. And therefore, however ſmall any given ſpace may be, a polygon may be inſcribed in the circle, and another deſcribed about it, each of which ſhall differ from the circle by a ſpace leſs than the given ſpace.

Cor. 2. The ſpace B which is greater than any polygon that can be inſcribed in the circle A, and leſs than any polygon that



can be deſcribed about it, is equal to the circle A. If not, let them be unequal; and firſt, let B exceed A by the ſpace C. Then, becauſe the polygons deſcribed about the circle A are all greater than B, by hypotheſis; and becauſe B is greater than A by the ſpace C, therefore, no polygon can be deſcribed about the circle A, but muſt exceed it by a ſpace greater than C, which is abſurd. In the ſame manner, if B be leſs than A by the ſpace C, it is ſhewn that no polygon can be inſcribed in the circle A, but is leſs than A by a ſpace greater than C, which is alſo abſurd. Therefore, A and B are not unequal, that is, they are equal to one another.

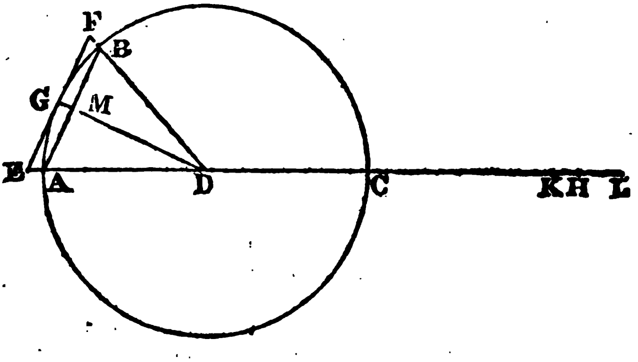
PROP.

PROP. V. THEOR.

EVERY circle is equal to the rectangle contained by the ſemidiameter, and a ſtraight line equal to half the circumference.

Let ABC be a circle, of which the centre is D, and the diameter AC; if in AC produced there be taken AH equal to half the circumference, the circle is equal to the rectangle contained by DA and AH.

Let AB be the ſide of any equilateral polygon in-
ſcribed in the circle ABC; biſect the circumference
AB in G, and through G draw EGF touching the circle,
and meeting DA produced in E, and DB produced in
F; EF will be the ſide of an equilateral polygon deſcribed
about the circle ABC [a]. In AC produced take AK equal to a 3. 8.
half the perimeter of the polygon whoſe ſide is AB; and AL
equal to half the perimeter of the polygon whoſe ſide is EF.
Then AK will be leſs, and AL greater than the ſtraight line
AH [b]. Now, becauſe in the triangle EDF, DG is drawn b Ax. 8.

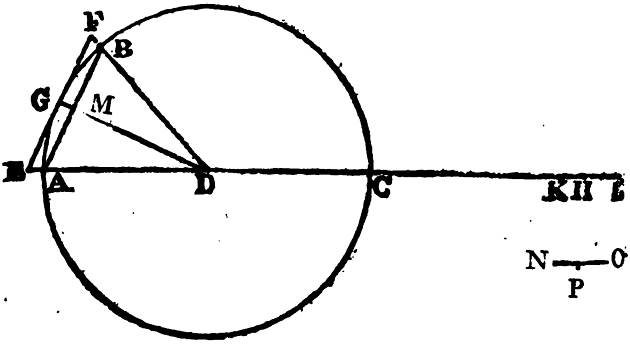


perpendicular to the baſe, the triangle EDF is equal to the rect-
ngle contained by DG and the half of EF [c]; and as the ſame c 41. 1.
s true of all the other equal triangles having their vertices
n D, which make up the polygon deſcribed about the
ircle; therefore, the whole polygon is equal to the rectangle

Book VIII. contained by DG and AL, half the perimeter of the polygon [d], or by DA and AL. But AL is greater than AH, therefore the rectangle DA, AL is greater than the rectangle DA, AH; the rectangle DA, AH is therefore leſs than the rectangle DA, AL, that is, than any polygon deſcribed about the circle ABC.

d 1, 2.

Again, the triangle ADB is equal to the rectangle contained by DM the perpendicular, and one half of the baſe AB, and it is therefore leſs than the rectangle contained by DG, or DA, and the half of AB. And as the ſame is true of all the other triangles having their vertices in D, which make up the inſcribed polygon, therefore the whole of the inſcribed polygon is leſs than the rectangle contained by DA, and AK half the perimeter of the polygon Now, the rectangle DA, AK is leſs than DA, AH; much more, therefore, is the polygon whoſe ſide is AB leſs than DA, AH; and the rectangle DA, AH is therefore greater than any polygon inſcribed in the circle ABC. But the ſame rectangle DA, AH has been proved to be leſs than any polygon deſcribed about the circle ABC; therefore, the rectangle DA, AH is equal to the circle ABC [e]. Now, DA is the ſemidiameter of the circle ABC, and AH the half of its circumference. Therefore, &c. Q. E. D.



e 2. cor. 4. 8.

Cor. 1. Hence, a polygon may be deſcribed about a circle, the perimeter of which ſhall exceed the circumference of the circle by a line that is leſs than any given line. Let NO be the given line. Take in NO the part NP leſs than its half, and leſs alſo than AD, and let a polygon be deſcribed about

out the circle ABC, ſo that its exceſs above ABC may be s than the ſquare of NP [f]. Let the ſide of this polygon be F. And ſince, as has been proved, the circle is equal to e rectangle DA, AH, and the polygon to the rectangle A, AL, the exceſs of the polygon above the circle is equal the rectangle DA, HL; therefore the rectangle DA, HL leſs than the ſquare of NP; and therefore, ſince DA is eater than NP, HL is leſs than NP, and twice HL leſs than ice NP, wherefore, much more is twice HL leſs than NO. t HL is the difference between half the perimeter of the lygon whoſe ſide is EF, and half the circumference of the cle; therefore, twice HL is the difference between the ole perimeter of the polygon and the whole circum-ence of the circle. The difference, therefore, between e perimeter of the polygon and the circumference of the cle is leſs than the given line NO.

Cor. 2. Hence alſo, a polygon may be inſcribed in a circle, h that the exceſs of the circumference above the perimeter the polygon may be leſs than any given line. This is oved like the preceding.

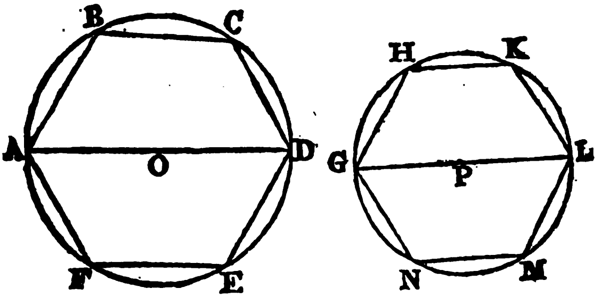
[f] 1. cor. 4. 8.

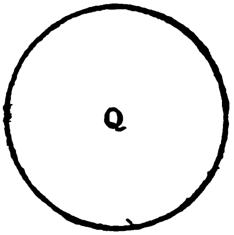
PROP. VI. THEOR.

CIRCLES are to one another in the duplicate ratio, or as the ſquares of their diameters.

Let ABD and GHL be two circles, of which the diameters are AD and GL; the circle ABD is to the circle GHL as the ſquare of AD to the ſquare of GL.

Let ABCDEF and GHKLMN be two equilateral polygons of the ſame number of ſides inſcribed in the circles ABD, GHL; and let Q be ſuch a ſpace that the ſquare of AD is to the ſquare of GL as the circle ABD to the ſpace Q.





Becauſe the polygons ABCDEF and GHKLMN are equilateral and of the ſame number of ſides, they are ſimilar [a], and are as the ſquares of the diameters of the circles in which they are inſcribed. Therefore, as the ſquare of AD to the ſquare of GI, ſo is the polygon ABCDEF to the polygon GHKLMN;

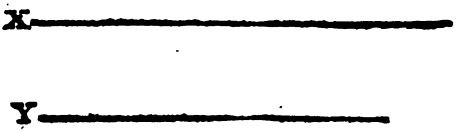
[a] 2. 8.

GHKLMN; but as the ſquare of AD to the ſquare of GL, ſo is the circle ABD to the ſpace Q; therefore, the polygon ABCDEF is to the polygon GHKLMN as the circle ABD to the ſpace Q. But the polygon ABCDEF is leſs than the circle ABD, therefore GHKLMN is leſs than the ſpace Q[b]; b 14. 7. wherefore, the ſpace Q is greater than any polygon inſcribed in the circle GHL.

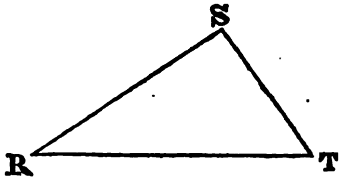
In the ſame manner it is demonſtrated, that Q is leſs than any polygon deſcribed about the circle GHL; wherefore, the ſpace Q is equal to the circle GHL[c]. Now, by hypotheſis, c 2. cor. 4. 8. the circle ABD is to the ſpace Q as the ſquare of AD to the ſquare of GL; therefore, the circle ABD is to the circle GHL as the ſquare of AD to the ſquare of GL. Therefore, &c. Q. E. D.

Cor. 1. Hence, the circumferences of circles are to one another as their diameters.

Let the ſtraight line X be equal to half the circumference of the circle ABD, and the ſtraight line Y to half the circumference of the circle GHL: And becauſe the rectangles AO, X, and CP, Y are equal to the circles ABD and GHL[d]; d 5. 8. therefore the rectangle AO, X is to the rectangle GP, Y as the ſquare of AD to the ſquare of GL, or as the ſquare of AO to the ſquare of GP. Therefore, alternately the rectangle AO, X is to the ſquare of AO, as the rectangle GP, Y to the ſquare of GP; but rectangles that have equal altitudes are as their baſes[e], therefore X is to e 1. 6. AO as Y to GP; and again, alternately, X is to Y as AO to GP, and taking the doubles of each, the circumference ABD is the circumference GHL as the diameter AD to the diameter GL.



Cor. 2. The circle that is deſcribed upon the ſide of a right angled triangle oppoſite to the right angle, is equal to the two circles deſcribed on the other two ſides. For the circle deſcribed upon SR is to the circle deſcribed upon RT as the ſquare of SR to the ſquare of RT; and the circle deſcribed upon TS is to the circle deſcribed upon RT as the ſquare of ST to the ſquare of RT. Wherefore, the circles deſcribed on SR and ST are to the circle deſcribed on RT as the ſquares of SR and ST to the
f 25. 5. ſquare of RT [f]. But the ſquares of RS and ST are equal to
g 47. 1. the ſquare of RT [g]; therefore the circles deſcribed on RS and ST are equal to the circle deſcribed on RT.

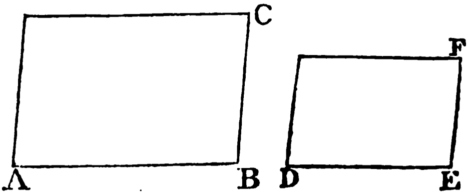


PROP. VII. THEOR.

EQUIANGULAR parallelograms are to one another as the products of the numbers proportional to their ſides.

Let AC and DF be two equiangular parallelograms, and let M, N, P and Q be four numbers, ſuch that AB : BC : : M : N; AB : DE : : M : P, and AB : EF : : M : Q, and therefore *ex æquali*, BC : EF : : N : Q. The parallelogram AC is to the parallelogram DF as MN to PQ.

Let NP be the product of N into P, and the ratio of MN to PQ will be compounded of the
a 10. def. 5. ratios [a] of MN to NP, and of NP to PQ. But the ratio of MN to
b 15. 5. NP is the ſame with that of M to P [b], becauſe MN and NP are equimultiples of M and P; and, for the ſame reaſon, the



ratio of NP to PQ is the ſame with that of N to Q; therefore, the ratio of MN to PQ is compounded of the ratios of M to P, and of N to Q. Now, the ratio of M to P is the ſame with that of the ſide AB to the ſide DE [c]; and the ratio c by Hyp. of N to Q the ſame with that of the ſide BC to the ſide EF. Therefore the ratio of MN to PQ is compounded of the ratios of AB to DE, and of BC to EF. And the ratio of the parallelogram AC to the parallelogram DF is compounded of the ſame ratios [d]; therefore, the parallelogram AC is to d 23. 6. the parallelogram DF as MN, the product of the numbers M and N, to PQ the product of the numbers P and Q. Therefore, &c. Q. E. D.

Cor. 1. Hence, if GH be to KL as the number M to the number N; the ſquare deſcribed on GH will be to the ſquare deſcribed on KL as MM the ſquare of the number M to NN the ſquare of the number N.

Cor. 2. If the ſquare on GH be to the ſquare on KL as a number to a number, the line GH will be to the line KL as the ſquare root of the firſt number to the ſquare root of the ſecond number. For, if GH were to KL in any other ratio, the ſquare deſcribed on GH would not be to the ſquare deſcribed on KL in the ratio ſuppoſed.

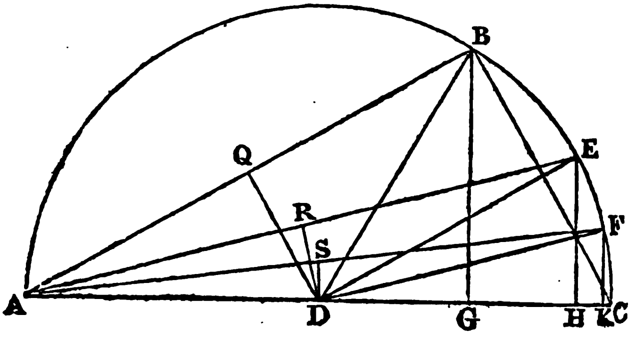
Nota. In the following Prop. the character + or — placed after a number ſignifies ſomething is to be added to that number, or ſomething ſubtracted from it. Thus, 1732.0508+ ſignifies a number greater than 1732.0508, &c. Alſo, it is to be obſerved, that the figure for diſtinctneſs is ſeparated into 2, the ſecond being a part of the firſt magnified or drawn on a greater ſcale.

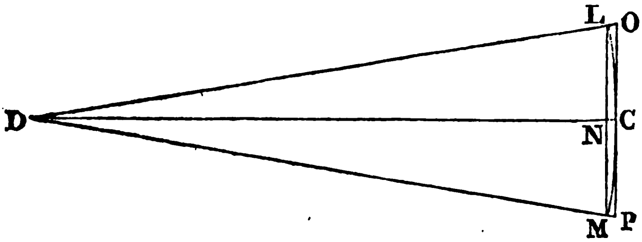
PROP. VIII. THEOR.

See N. THE circumference of a circle exceeds three times the diameter by a part leſs than one ſeventh of the diameter, but greater than ten of the parts whereof the diameter contains ſeventy-one.

Let ABC be a circle, of which the centre is D, and the diameter AC; the circumference is greater than three times the diameter AC by a line that is leſs than $\frac{1}{7}$ of AC and greater than $\frac{10}{71}$ of AC.

In the circle apply the ſtraight line CB equal to CD. Biſect CB in E, CE in F, and ſo on till there be five biſections, and let CL, (repreſented in the 2d figure) be the arch found by the fifth biſection. Take the arch CM equal to CL; join LM,





LM, and through C draw OCP touching the circle in C, and meeting DL, DM produced in O and P. Join (fig. 1.) AB, AE, AF, and alſo DB, DE, DF; draw BG, EH, FK perpendicular to DC, and DQ, DR, DS perpendicular to AB, AE, AF.

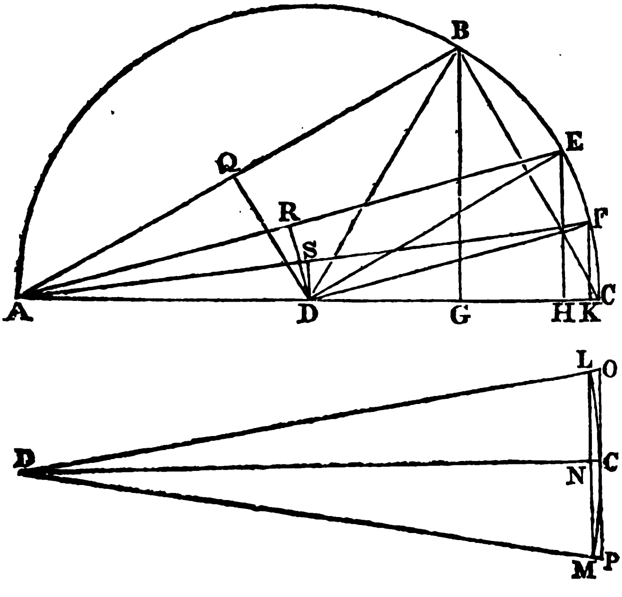
Becauſe the ſtraight line CB is equal to CD, the ſemidiameter of the circle, the arch CB is the ſixth part of the circumference [a]; therefore, CE is the 12th part, CF the 24th part; the half of CF the 48th; the half of that half the 96th, and the half of the 96th is that to which CL is equal. But CM is equal to CL; and therefore, the arch LM is alſo the 96th part of the circumference, ſo that the ſtraight line LM is the ſide of an equilateral polygon of 96 ſides inſcribed in the circle, and OP the ſide of an equilateral polygon of the ſame number of ſides deſcribed about the circle [b]. Now, becauſe BD is equal to BC, the angle BDC is equal to the angle BCD, and the angles BGD, BGC are alſo equal, being right angles; and the ſide BG is common to both the triangles ADG, ACG; therefore the baſe DG is equal to the baſe GC [c], and DG is the half of DC or AD. Therefore, DG is to DA as 1 to 2; and, by compoſition, AG to AD as 3 to 2: Now, AC is to AD alſo as 2 to 1; therefore, the rectangle CA, AG is to the ſquare of DA as 6 to 2, or as 3 to 1 [d]. But, becauſe the angle ABC is a right angle, the rectangle CA, AG is equal to the ſquare of AB [e]; therefore, the ſquare of AB is to the ſquare of AD as 3 to 1; or as 3000000 to 1000000; and therefore, AB is to AD as 1732.0508+ to 1000 [f], for the ſquare of 1732.0508 is leſs than 3000000, and the ſquare of 1000 is equal to 1000000. But AQ is the half of AB, becauſe DQ is drawn from the centre perpendicular to AB, therefore alſo, AQ is to AD as 866.0254+ to 1000.

And becauſe the arch CE is the half of the arch CB, the angle CDE is half of the angle CDB [g]; and the angle DAB is alſo the half of CDB [h], therefore the angles CDE, DAB are equal. In the triangles DAQ, EDH, therefore, the angle DAQ is equal to the angle EDH, and the angle AQD to the angle DHE, becauſe they are both right angles; and the ſide AD is alſo equal to the ſide DE, therefore the triangle DAQ is equal to the triangle EDH, and the ſide AQ to the ſide DH [c]. And it has been ſhewn, that AQ is to AD as 866.0254+ to 1000, therefore alſo DH is to AD as

866.0254

[a] 14. 4.
[b] 3. 8.
[c] 26. 1.
[d] 7. 8.
[e] 8. 6.
[f] 2. cor. 7. 8.
[g] 36. 6.
[h] 20. 3.

866.0254+ to 1000; and, by compoſition, AH to AD as
1866.0254+ to 1000. Therefore alſo, the rectangle CA,
d 7. 8. AH is to the ſquare of AD[d] as $\overline{2000} \times \overline{1866.0254+}$ to
$\overline{1000} \times \overline{1000}$, that is, as 3732050.8+ to 1000000. Wherefore alſo, the ſquare of AE, which is equal to the rectangle
e 8. 6. CA, AH[e], is to the ſquare of AD as 3732050.8+ to
1000000, and therefore, AE is to AD as 1931.8516+ to
1000, for the ſquare of 1931.8516 is leſs than 3732050.8.
Therefore, AR, which is the half of AE, or DK, which is
ſhewn to be equal to AR in the ſame manner that DH was
proved equal to AQ, is to AD as 965.9258+ to 1000.



Hence again, by compoſition, AK is to AD as 1965.9258+ to 1000, and the rectangle CA, AK, or the ſquare of AF, to the ſquare of AD as 3931851.6+ to 1000000[d], and AF to AD as 1982.8897 + to 1000, for the ſquare of 1982.8897 is leſs than 3931851.6.

In the ſame way it will be ſhewn, that the ſtraight line drawn from A to the third point of biſection, is to AD as 1995.7178+ to 1000; and therefore, that the ſquare of the

line

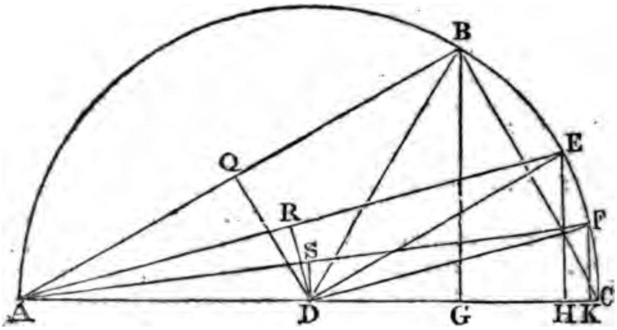
ine drawn from A to the fourth point of biſection, is to the quare of AD as 3995717.8+ to 1000000. But DN (fig 2.) s equal to half the line from A to the fourth point of biſection, for L is the fifth point of biſection, and therefore the quare of DN is the fourth part of the ſquare of that line; wherefore, the ſquare of DN is to the ſquare of AD, or of DC as 998929.45+ to 1000000. Therefore, by diviſion, the quare of LN, which is the exceſs of the ſquare of DC, or f DL, above the ſquare of DN, is to the ſquare of CD as he exceſs of 1000000 above 998929.45+ is to 1000000, that is, as 1070.55— to 1000000. Therefore alſo, LN has to CD the ratio of 32.71927— to 1000; for the ſquare of 32.71927 is greater than 1070.55.

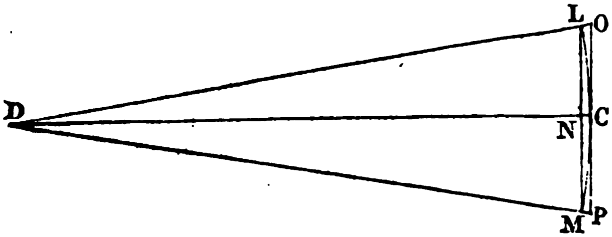
Now, ſince DN is to DC as 999.46455+ to 1000; and LN to DC as 32.71927— to 1000; therefore, *ex equo*, DN is to LN as 999.46455+ to 32.71927—. But, as DN to NL, ſo DC to CO; therefore, DC is to CO as 999.46455+ to 32.71927—. Now, the ratio of 999.46455+ to 32.71927— is greater than that of 999.46455 to 32.71927[l]; wherefore l 8. 5.
DC has to CO a greater ratio than 999.46455 has to 32.71927, and conſequently, CO to DC a leſs ratio than 32.71927 to 999.46455. But the ratio of 32.7367 to 1000 is greater than the ratio of 32.71927 to 999.46455; therefore, the ratio of CO to DC is alſo leſs than that of 32.7367 to 1000[m]. And 96 times CO is half the perimeter of the cir- m 13. 5.
cumſcribed polygon; therefore, the ratio of half the perimeter of that polygon to the ſemidiameter DC, or the ratio of the whole perimeter to the whole diameter AC, is leſs than the ratio of $\overline{32.7367} \times \overline{96}$ to 1000, or than the ratio of 3142.7232 to 1000. But the circumference of the circle ABC is leſs than the perimeter of the polygon of which the ſide is CO; therefore, the circumference of the circle has to the diameter a leſs ratio than 3142.7232 has to 1000. Now, the ratio of 22 to 7 is greater than that of 3142.7232 to 1000, therefore, the circumference of the circle ABC has to the diameter a leſs ratio than 22 has to 7; and therefore, the part by which the circumference exceeds three times the diameter is leſs than a ſeventh of the diameter.

It remains to demonſtrate, that the part by which the circumference exceeds three times the diameter is greater than $\frac{10}{71}$ of the diameter.

Becauſ

Becauſe, as was already ſhewn, the ſquare of AB is to the ſquare of AD as 3000000 to 1000000, AB is to AD as 1732.0509— to 1000; for the ſquare of 1732.0509 is greater than 3000000. Therefore, the half of AB, that is, AQ or DH is to DA as 866.02545— to 1000, and, by compoſition, AH to AD as 1866.02545— to 1000; and therefore, the rectangle CA, AH, that is, the ſquare of AE is to the ſquare of AD as 3732050.9— to 1000000; and therefore, alſo, AE is to AD as 1931.8517— to 1000; for the ſquare of 1931.8517 is greater than 3732050.9. Therefore alſo, AR, orDK, which is equal to AR, is to AD as 965.92585— to 1000, and, by compoſition, AK is to AD as 1965.92585—to 1000.





Hence, the rectangle CA, AK, that is, the ſquare of AF is to the ſquare of AD as 3931851.7— to 1000000; and therefore, AF to AD as 1982.8898— to 1000, for the ſquare of 1982.8898 is greater than 3931851.7.

In like manner it is ſhewn, that the ſtraight line drawn from A to the third point of biſection is to AD as 1995.7179—

to 1000; and that, therefore, the ſquare of the line drawn from A to the fourth point of biſection, is to the ſquare of AD as 3995717.9— to 1000000. But DN is the half of the ſtraight line drawn from A to the fourth point of biſection, becauſe L is the fifth point of biſection, and therefore the ſquare of DN is the fourth part of the ſquare of the line drawn to the point of the fourth biſection; therefore the ſquare of DN is to the ſquare of AD, or of DC, as 998929.475— to 1000000. Wherefore alſo, the exceſs of the ſquare of DC or of DL above the ſquare of DN, that is, the ſquare of LN is to the ſquare of DC as the exceſs of 1000000 above 998929.475—, or as 1070.525+ to 1000000; and therefore, LN is to DC as 32.71887+ to 1000, for the ſquare of 32.71887 is leſs than 1070.525. Therefore, LN has to DC a greater ratio[1] than 32.71887 to 1000, and 96 times LN, or half the perimeter of the inſcribed polygon of 96 ſides, has to the ſemidiameter DC a greater ratio than that of $\overline{32.71887 \times 96}$ to 1000; therefore alſo, the whole perimeter of the polygon has to the whole diameter AC a greater ratio than $\overline{32.71887 \times 96}$, or than 3141.01252 has to 1000. Now, the circumference of the circle is greater than the perimeter of the inſcribed polygon, therefore the circumference of the circle has to the diameter a greater ratio than 3141.01252 has to 1000. But the ratio of 3141.01252 to 1000 is greater than that of 3 and $\frac{10}{71}$ to 1. Wherefore, the exceſs of the circumference above three times the diameter is greater than 10 of thoſe parts whereof the diameter contains 71; and it has already been ſhewn to be leſs than one ſeventh of the diameter. Therefore, &c. Q. E. D.

[1] 8. 5.

Cor. 1. Hence, the diameter of a circle being given, the circumference may be ſound nearly, by making as 7 to 22, ſo the diameter to a fourth proportional, which will be greater than the circumference. And, if as 1 to 3 and $\frac{10}{71}$ ſo the diameter to a fourth proportional, this will alſo be nearly equal to the circumference, but will be leſs than it.

Cor. 2. Becauſe the difference between $\frac{1}{7}$ and $\frac{10}{71}$ is $\frac{1}{497}$, therefore the lines found by theſe proportions differ by $\frac{1}{497}$ of the diameter. Therefore, the difference of either of them from the circumference muſt be leſs than the 497th part of the diameter.

Cor. 3. The ratio of a circle to the ſquare of its diameter is nearly that of 11 to 14. For the ratio of the circumference to the diameter is that of 22 to 7 nearly, and therefore, the ratio of half the circumference to the diameter, is nearly that of 11 to 7; and the ratio of half the diameter to the whole is that of 1 to 2; therefore, the ratio compounded of theſe, that is, of the rectangle under half the circumference and half the diameter, to the ſquare of the diameter is that of 11 to 14, nearly. But the rectangle under half the circumference and half the diameter is equal to the circle [n]; therefore, the circle is to the ſquare of the diameter as 11 to 14 nearly. n 5. 8.

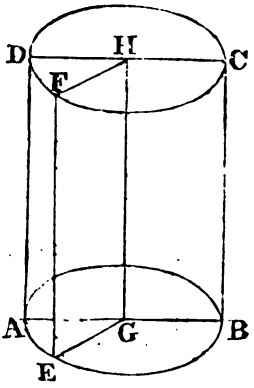
Cor. 4. It is evident, that the method uſed in this propoſition for finding the limits of the ratio of the circumference to the diameter may be carried to a greater degree of exactneſs, by finding the perimeter of an inſcribed, and a circumſcribed polygon of a greater number of ſides than 96. Thus by inſcribing in the circle a polygon of 40960 ſides, and deſcribing about it another of the ſame number of ſides, it may be demonſtrated, that the ratio of the circumference to the diameter is greater than that of 314159265 to 100000000, but leſs than that of 314159266 to 100000000.

PROP.

PROP. IX. THEOR.

IF from any point in the circumference of the baſe of a cylinder a ſtraight line be drawn perpendicular to the plane of the baſe, it will be wholly in the cylindric ſuperficies.

Let ABCD be a cylinder, of which the baſe is the circle AEB, DFC the circle oppoſite to the baſe, and GH the axis; from E, any point in the circumference AEB, let EF be drawn perpendicular to the plane of the circle AEB; the ſtraight line EF is in the ſuperficies of the cylinder.



Let F be the point in which EF meets the plane DFC oppoſite to the baſe; join EG and FH; and let AGHD be the rectangle[a] by the revolution of which the cylinder ABCD is deſcribed. a 1. def. 7.

Now, becauſe GH is at right angles to GA, the ſtraight line which by its revolution deſcribes the circle AEB, it is at right angles to all the ſtraight lines in the plane of that circle which meet it in G, and it is therefore at right angles to the plane of the circle AEB. But EF is at right angles to the ſame plane; therefore, EF and GH are parallel[b], and in the ſame plane. And ſince the plane through GH and EF cuts the parallel planes AEB, DEC in the ſtraight lines EG and FH, EG is parallel to FH[c]. The figure EGHF is therefore a parallelogram, and it has the angle EGH a right angle, therefore it is a rectangle, and is equal to the rectangle AH, becauſe EG is equal to AG. Therefore, when in the revolution of the rectangle AH, the ſtraight line AG coincides with EG, the two rectangles AH and EH will coincide, and the ſtraight line AD will coincide

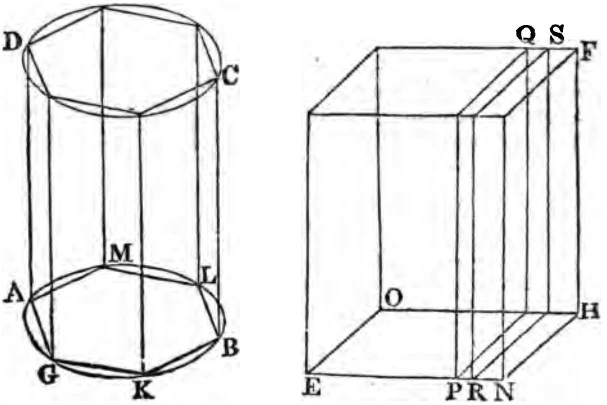
b 6. 7.
c 14. 7.

incide with the ſtraight line EF. But AD is always in the ſuperficies of the cylinder, for it deſcribes that ſuperficies; therefore, EF is alſo in the ſuperficies of the cylinder. Therefore, &c. Q. E. D.

PROP. X. THEOR.

A CYLINDER and a parallelepiped having equal baſes and altitudes are equal to one another.

Let ABCD be a cylinder, and EF a parallelepiped havin equal baſes, viz. the circle AGB and the parallelogram EH and having alſo equal altitudes; the cylinder ABCD is equa to the parallelepiped EF.



If not, let them be unequal; and firſt, let the cylinder b leſs than the parallelepiped EF; and from the parallelepipe EF let there be cut off a part EQ by a plane PQ parallel t NF, equal to the cylinder ABCD. In the circle AGB in ſcribe the polygon AGKBLM that ſhall differ from the circl by a ſpace leſs than the parallelogram PH, and cut off fro the parallelogram EH, a part OR equal to the polygo AGKBLM. The point R will fall between P and N. O the polygon AGKBLM let an upright priſm AGBCD be con ſtituted of the ſame altitude with the cylinder, which wi therefore be leſs than the cylinder becauſe it is within it [b]; and if through the point R a plane RS parallel to NF be nade to paſs, it will cut off the parallelepiped ES equal to[b] he priſm AGBC, becauſe its baſe is equal to that of the riſm, and its altitude is the ſame. But the priſm AGBC is eſs than the cylinder ABCD, and the cylinder ABCD is e-ual to the parallelepiped EQ, by hypotheſis; therefore, ES leſs than EQ, and it is alſo greater, which is impoſſible. The cylinder ABCD, therefore, is not leſs than the parallelepiped EF; and in the ſame manner it may be ſhewn ot to be greater than EF. Therefore they are equal. Q. E. D.

[b] 9. 8.

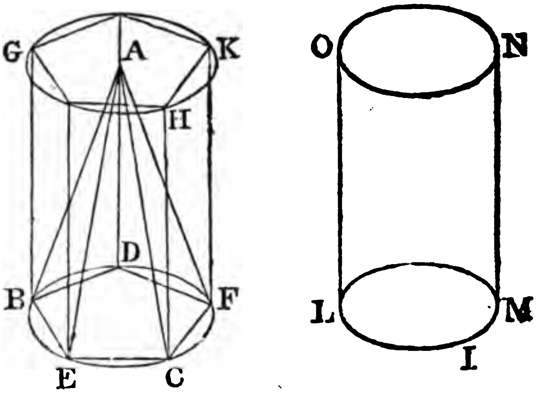
T P R P.

PROP. XI. THEOR.

IF a cone and a cylinder have the ſame baſe and the ſame altitude, the cone is the third part of the cylinder.

Let the cone ABCD, and the cylinder BFKG have th ſame baſe, viz. the circle BCD, and the ſame altitude, viz. the perpendicular from the point A upon the plane BCD, th cone ABCD is the third part of the cylinder BFKG.

If not, let the cone ABCD be the third part of another cylinder LMNO, having the ſame altitude with the cylinder BFKG, but let the baſes BCD and LIM be unequal; and firſt, let BCD be greater than LIM.



Then becauſe the circle BCD is greater than the circl LIM, a polygon may be inſcribed in BCD, that ſhall diffe from it leſs than LIM does,[a] and which, therefore, will b greater than LIM. Let this be the polygon BECFD and upon BECFD let there be conſtituted the pyrami ABECFD, and the priſm BCFKHG.

[a] 4. 8.

Becauſe the polygon BECFD is greater than the circl LIM, the priſm BCFKHG, is greater than the cylinde

LMNO, for they have the ſame altitude, but the priſm has the greater baſe. But the pyramid ABECFD is the third part of the priſm [b] BCFKHG, therefore, it is greater than the third part of the cylinder LMNO. Now, the cone ABECFD is, by hypotheſis, the third part of the cylinder LMNO, therefore, the pyramid ABECFD is greater than the cone ABCD, and it is alſo leſs, becauſe it is inſcribed in the cone, which is impoſſible. Therefore, the cone ABCD is not leſs than the third part of the cylinder BFKG: And, in the ſame manner, by circumſcribing a polygon about the circle BCD, it may be ſhewn, that the cone ABCD is not greater than the third part of the cylinder BFKG; therefore, it is equal to the third part of that cylinder. Q. E. D.

b 35. 7.

PROP. XII. THEOR.

IF a hemiſphere and a cone have equal baſes and altitudes, a ſeries of cylinders may be inſcribed in the hemiſphere, and another ſeries may be deſcribed about the cone, having all the ſame altitudes with one another, and ſuch that their ſum ſhall differ from the ſum of the hemiſphere, and the cone by a ſolid leſs than any given ſolid.

Let ADB be a ſemicircle, of which the centre is C, and let CD be at right angles to AB; let DB and DA be ſquares deſcribed on DC, draw DE and let the figure thus conſtructed revolve about DC: then, the ſector BCD, which is the half of the ſemicircle ADB, will deſcribe a hemiſphere having C for its centre [a], and the triangle CDE will deſcribe a cone, having its vertex at C, and having for its baſe the circle [b] deſcribed by DE, equal to that deſcribed by BC, which is the baſe of the hemiſphere. Let W be any given ſolid. A ſeries of cylinders may be inſcribed in the hemiſphere ADB, and another deſcribed about the cone ECL, ſo that their ſum ſhall differ from the ſum of the hemiſphere and the cone, by a ſolid leſs than the ſolid W.

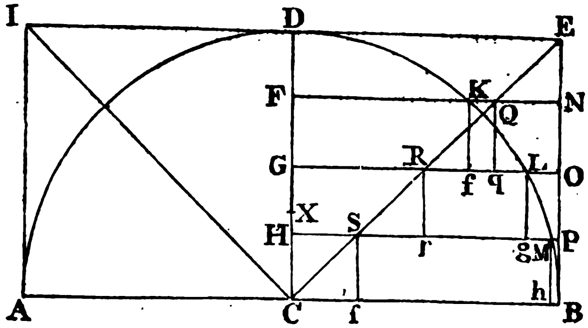
a 14. def. 7.

b 18. def. 7.

Upon the baſe of the hemiſphere let a cylinder be conſtituted equal to W, and let its altitude be CX. Divide CD into ſuch a number of equal parts, that each of them ſhall be leſs than CX; let theſe be CH, HG, GF and FD. Through the points F, G, H, draw FN, GO, HP parallel to CB, meeting the circle in the points K, L and M; and the ſtraight line CE in the points Q, R and S. From the points K, L, M draw Kf, Lg, Mh perpendicular to GO, HP and CB; and from Q, R and S, draw Qq, Rr, Ss perpendicular to the ſame lines. It is evident, that the figure being thus conſtructed, if the whole revolve about CD, the rectangles Ff, Gg, Hh will deſcribe cylinders [c] that will be circumſcribed by the hemiſphere BDA; and that the rectangles DN, Fq, Gr, Hs will alſo deſcribe cylinders that will circumſcribe the cone ICE. Now, it may be demonſtrated, as was done of the priſms inſcribed in a pyramid [d], that the ſum of all the cylinders deſcribed within the hemiſphere, is exceeded by the hemiſphere by a ſolid leſs than the cylinder generated by the rectangle HB, that is, by

c 20. def. 7.

d 33. 7.



a ſolid leſs than W, for the cylinder generated by HB is leſs than W. In the ſame manner, it may be demonſtrated, that the ſum of the cylinders circumſcribing the cone ICE is greater than the cone by a ſolid leſs than the cylinder generated by the rectangle DN, that is, by a ſolid leſs than W. Therefore, ſince the ſum of the cylinders inſcribed in the hemiſphere, together with a ſolid leſs than W, is equal to the hemiſphere; and, ſince the ſum of the cylinders deſcribed about the cone is equal to the cone together with a ſolid leſs than W; adding equals to equals, the ſum of all theſe cylinders, together with a ſolid leſs than W, is equal to the ſum of the hemiſphere and the cone together with a ſolid leſs than W. Therefore, the difference between the whole of the cylinders and the ſum of the hemiſphere and the cone, is equal to the difference of two ſolids, which are each of them leſs than W; but this difference muſt alſo be leſs than W, therefore the difference between the two ſeries of cylinders and the ſum of the hemiſphere and cone is leſs than the given ſolid W. Q. E. D. Book VIII.

PROP. XIII.

THE ſame things being ſuppoſed as in the laſt propoſition, the ſum of all the cylinders inſcribed in the hemiſphere, and deſcribed about the cone, is equal to a cylinder, having the ſame baſe and altitude with the hemiſphere.

Let the figure DCB be conſtructed as before, and ſuppoſed to revolve about CD; the cylinders inſcribed in the hemiſphere, that is, the cylinders deſcribed by the revolution of the rectangles Hh, Gg, Ff, together with thoſe deſcribed about the cone, that is, the cylinders deſcribed by the revolution of the rectangles Hs, Gr, Fq, and DN, are equal to the cylinder deſcribed by the revolution of the rectangle DB.

Let L be the point in which GO meets the circle ADB, then, becauſe CGL is a right angle if CL be joined, the circles deſcribed with the diſtances CG and GL are equal to the circle deſcribed with the diſtance CL[a] or GO; now CG is equal to GR, becauſe CD is equal to DE, and therefore alſo, the circles deſcribed with the diſtances GR and GL are together equal to the circle deſcribed with the diſtance GO, that is, the circles deſcribed by the revolution of GR and GL about the point G, are together equal to the circle deſcribed by the revolution of GO about the ſame point G; therefore alſo, the cylinders that ſtand upon the two firſt of theſe circles having the common altitude GH, are equal to the cylinder which ſtands on the remaining circle, andwhich has the ſame altitude GH. The cylinders deſcribed by the revolution of the rectangles Gg and Gr are therefore equal to the cylinder deſcribed by the rectangle GP. And as the ſame may be ſhewn of all the reſt, therefore the cylinders deſcribed by the rectangles Hh, Gg, Ff, and by the rectangles Hs, Gr, Fq, DN, are together equal to the cylinder deſcribed by DB, that is, to the cylinder having the ſame baſe and altitude with the hemiſphere. Q. E. D.

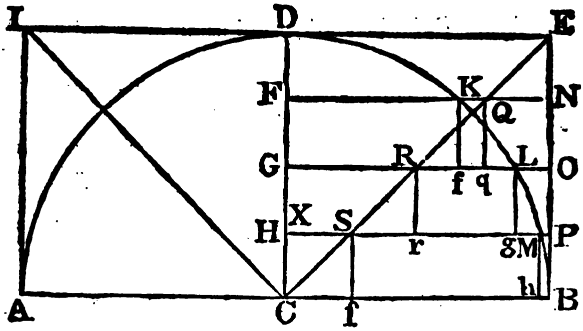
[a] 2. cor. 6. 8.

PROP.

PROP. XIV.

EVERY ſphere is two thirds of the circumſcribing cylinder.

Let the figure be conſtructed as in the two laſt propoſitions, and if the hemiſphere deſcribed by BDC be not equal to two thirds of the cylinder deſcribed by BD, let it be greater by the ſolid W. Then, as the cone deſcribed by CDE is one a 10. 8. third of the cylinder [a] deſcribed by BD, the cone and the hemiſphere together will exceed the cylinder by W. But



b 13. 8. that cylinder is equal to the ſum of all the cylinders deſcribed by the rectangles Hh, Gg, Ff, Hs, Gr, Fq, DN[b]; therefore the hemiſphere and the cone added together exceed the ſum of all theſe cylinders by the given ſolid W; c 12. 8. which is abſurd, for it has been ſhewn [c], that the hemiſphere and the cone together differ from the ſum of the cylinders by a ſolid leſs than W. The hemiſphere is therefore equal to two thirds of the cylinder deſcribed by the rectangle BD; and therefore the whole ſphere is equal to two thirds of the cylinder deſcribed by twice the rectangle BD, that is, to two thirds of the circumſcribing cylinder. Q. E. D.

FINIS.

ELEMENTS

OF

PLANE

AND

SPHERICAL

TRIGONOMETRY.

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M,DCC,XCV.

ELEMENTS

OF

PLANE TRIGONOMETRY.

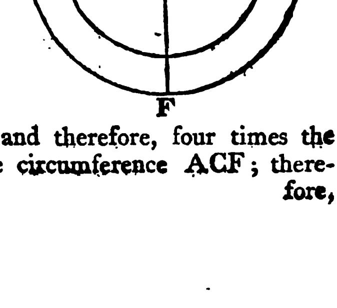
LEMMA I.

AN angle at the centre of a circle is to ſour right angles as the arch on which it ſtands is to the whole circumference.

Let ABC be an angle at the centre of the circle ACF, ſtanding on the circumference AC: the angle ABC is to four right angles as the arch AC to the whole circumference ACF.

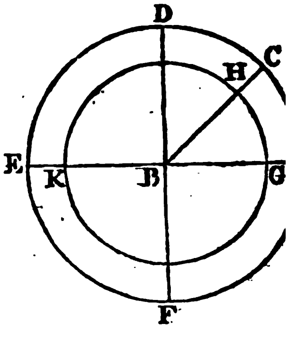
Produce AB till it meet the circle in E, and draw DBF perpendicular to AE.

Then, becauſe ABC, ABD are two angles at the centre of the circle ACF, the angle ABC is to the angle ABD as the arch AC to the arch AD, (36. 6.); and therefore alſo, the angle ABC is to four times the angle ABD as the arch AC to four times the arch AD (4. 5.).



But ABD is a right angle, and therefore, four times the arch AD is equal to the whole circumference ACF; therefore, the angle ABC is to four right angles as the arch A the whole circumference ACF.

Cor. Equal angles at the centre of different circles ſta arches which have the ſame ratio to their circumferences. For, if the angle ABC, at the centre of the circles ACE, GHK, ſtand on the arches AC, GH, AC is to the whole circumference of the circle ACE, as the angle ABC to four right angles; and the arch HG is to the whole circumference of the circle GHK in the ſame ratio. Therefore, &c.



DEFINITIONS.

I.

IF two ſtraight lines interſect one another in the centre circle, the arch of the circumference intercepted bet them is called the meaſure of the angle which they tain. Thus, the arch AC is the meaſure of the ABC.

II.

If the circumference of a circle be divided into 360 parts, each of theſe parts is called a degree; and, if gree be divided into 60 equal parts, each of theſe is cal minute; and, if a minute be divided into 60 equal parts, of them is called a ſecond, and ſo on. And as man grees, minutes, ſeconds, &c. as are in any arch, ſo degrees, minutes, ſeconds, &c. is ſaid to be in the angle ſured by that arch.

Cor. 1. Any arch is to the whole circumference of it is a part, as the number of degrees, and parts of a d contained in it, is to the number 360. And any angle four right angles as the number of degrees and parts of a degree in the arch, which is themeaſure of that angle, is to 360.

Cor. 2. Hence alſo, the arches which meaſure the ſame angle, whatever be the radii with which they are deſcribed, contain the ſame number of degrees, and parts of a degree. For the number of degrees and parts of a degree contained in each of theſe arches has the ſame ratio to the number 360, that the angle which they meaſure has to four right angles (Cor. Lem. 1.).

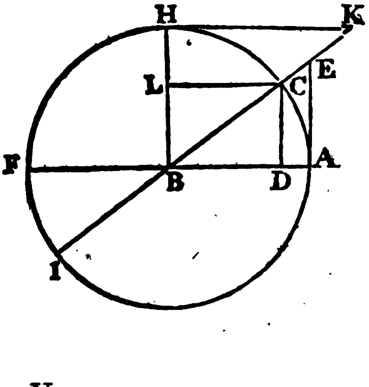
The degrees, minutes, ſeconds, &c. contained in any arch or angle, are uſually written as in this example, 49°. 36′. 24″. 42‴; that is, 49 degrees, 36 minutes, 24 ſeconds, and 42 thirds.

III.

Two angles which are together equal to two right angles, or two arches which are together equal to a ſemicircle, are called the ſupplements of one another.

IV.

A ſtraight line CD drawn through C, one of the extremities of the arch AC perpendicular to the diameter paſſing through the other extremity A, is called the *Sine* of the arch AC, or of the angle ABC, of which AC is the meaſure.



Cor. The ſine of a quadrant, or of a right angle, is equal to the radius.

V.

The ſegment DA of the diameter paſſing through A, one extremity of the arch AC, between the ſine CD, and that extremity is called the *Verſed ſine* of the arch AC, or of the angle ABC.

VI.

A ſtraight line AE touching the circle at A, one extremity of the arch AC, and meeting the diameter BC paſſing through the other extremity C in E, is called the *Tangent* of the arch AC, or of the angle ABC.

Cor. The tangent of half a right angle is equal to the radius.

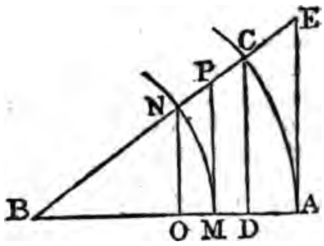
VII.

The ſtraight line BE between the centre and the extremity of the tangent AE, is called the *Secant* of the arch AC, or of the angle ABC.

Cor. to Def. 4. 6. 7. the ſine, tangent, and ſecant of any angle ABC, are likewiſe the ſine, tangent, and ſecant of its ſupplement CBF.

It is manifeſt from Def. 4. that CD is the ſine of the angle CBF. Let CB be produced till it meet the circle again in I; and it is manifeſt, that AE is the tangent, and BE the ſecant, of the angle ABI or CBF, from def. 8. 7.

Cor. to Def. 4. 5. 6. 7. The ſine, verſed ſine, tangent, and ſecant of an arch, which is the meaſure of any given angle ABC is to the ſine, verſed ſine, tangent and ſecant, of any other arch which is the meaſure of the ſame angle, as the radius of the firſt arch is to the radius of the ſecond.

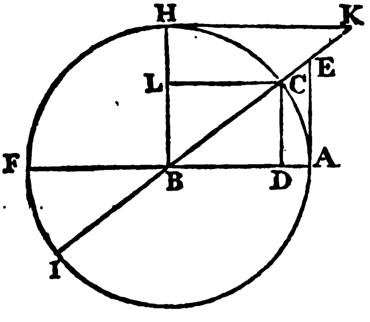


Let AC, MN be meaſures of the angle ABC, according to Def. 1. CD the ſine, DA the verſed ſine, AE the tangent, and BE the ſecant of the arch AC, according to def. 4. 5. 6. 7. and NO the ſine, OM the verſed ſine, MP the tangent, and BP the ſecant of the arch MN, according to the ſame definitions. Since CD, NO, AE, MP are parallel, CD is to NO as the radius CB to the radius NB, and AE to MP as AB to BM. and BC or BA to BD as BN or BM to BO; and, by converſion, DA to MO as AB to MB. Hence the corollary is manifeſt. And therefore, if tables be conſtructed, exhibiting in numbers the ſines, tangents, ſecants, and verſed ſines of certain angles to a given radius, they will exhibit the ratios of the ſines, tangents, &c. of the ſame angles to any radius whatſoever.

In ſuch tables, which are called trigonometrical tables, the radius is either ſuppoſed 1, or ſome number in the ſeries 10, 100, 1000, &c. The uſe and conſtruction of theſe tables will appear from what follows.

VIII.

The difference between any angle and a right angle, or between any arch and a quadrant, is called the complement of that angle, or of that arch. Thus, if BH be perpendicular to AB, the angle CBH is the complement of the angle ABC, and the arch HC the complement of AC; alſo the complement of the obtuſe angle FBC is the angle HBC, its exceſs above a right angle; and the complement of the arch FC is HC.



IX.

The ſine, tangent, or ſecant of the complement of any angle is called the coſine, cotangent, or coſecant of that angle. Thus, let CL or DB which is equal to CL, be the ſine of the angle CBH; HK the tangent, and BK the ſecant of the ſame angle; CL or BD is the coſine, HK the cotangent, and BK the coſecant of the angle ABC.

Cor. 1. The radius is a mean proportional between the tangent and co-tangent of any angle ABC.

For, ſince HK, BA are parallel, the angles HKB, ABC are equal, and KHB, BAE are right angles; therefore the triangles BAE, KHB are ſimilar, and therefore AE is to AB, as BH or BA to HK.

Cor. 2. The radius is a mean proportional between the coſine and ſecant of any angle ABC.

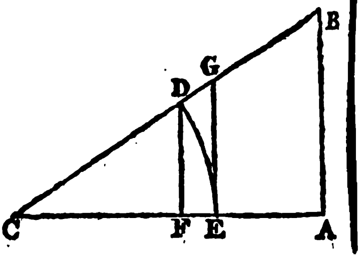
Since CD, AE are parallel, BD is to BC or BA, as BA to BE.

PROP. I.

IN a right angled plain triangle, as the hypotenuſe to either of the ſides, ſo the radius to the ſine of the angle oppoſite to that ſide; and as either of the ſides is to the other ſide, ſo is the radius to the tangent of the angle oppoſite to that ſide.

Let ABC be a right angled plane triangle, of which BC is the hypotenuſe. From the centre C, with any radius CD, deſcribe the arch DE; draw DF at right angles to CE, and from E draw EG touching the circle in E, and meeting CB in G: DF is the ſine, and EG the tangent of the arch DE, or of the angle C.

The two triangles DFC, BAC are equiangular, becauſe the angles DFC, BAC are right angles, and the angle at C is common. Therefore, CB : BA : : CD : DF; but CD is the radius, and DF the ſine of the angle C (def. 4.), therefore CB : BA : : R : ſin. C.

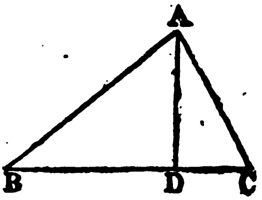


Alſo, becauſe EG touches the circle in E, CEG is a right angle, and therefore equal to the angle BAC; and ſince the angle at C is common to the triangles

CBA, CGE, theſe triangles are equiangular, wherefore CE : EG : : CA : AB ; but CE is the radius, and EG the tangent of the angle C ; therefore, CA : AB : : R : tan. C.

Cor. 1. As the radius to the ſecant of the angle C, ſo the ſide adjacent to that angle to the hypotenuſe. For CG is the ſecant of the angle C (def. 7.), and the triangles CGE, CBA being equiangular, CA : CB : : CE : CG, that is, CA : CB : : R : ſec. C.

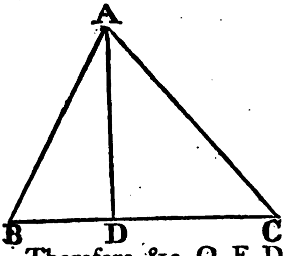
Cor. 2. In every triangle, if a perpendicular be drawn from any of the angles on the oppoſite ſide, the ſegments of that ſide are to one another, as the tangents of the parts into which the oppoſite angle is divided by the perpendicular. For, if in the triangle ABC, AD be drawn perpendicular to the baſe BC, each of the triangles CAD, ABD being right angled, AD : DC : : R : tan. CAD, and AD : DB : R : tan. DAB ; therefore, *ex æquo*, DC : DB : : tan. CAD : tan. BAD.



PROP. II.

THE ſides of a plain triangle are to one another as the ſines of the oppoſite angles.

From A any angle in the triangle ABC, let AD be drawn perpendicular to BC. And becauſe the triangle ABD is right angled at D, AB : AD : : R : ſin. B ; and, for the ſame reaſon, AC : AD : : R : ſin. C, and inverſely, AD : AC : : ſin. C : R ; therefore, *ex æquo perturbatè*, AB : AC : : ſin. C : ſin. B. In the ſame manner, it may be demonſtrated, that AB : BC : : ſin. C : ſin. A. Therefore, &c. Q. E. D.

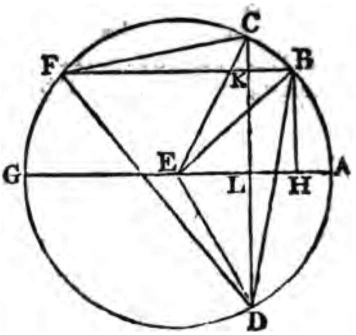


PROP. III.

THE ſum of the ſines of any two arches of a circle, is to the difference of their ſines, as the tangent of half the ſum of the arches to the tangent of half their difference.

Let AB, AC be two arches of a circle ABCD; let E be the centre, and AEG the diameter which paſſes through A. Draw BF parallel to AG, meeting the circle again in F. Draw BH and CL perpendicular to AE, and they will be the ſines of the arches AB and AC; produce CL till it meet the circle again in D; join DF, FC, DE, EB, EC.

Now, ſince EL from the centre is perpendicular to CD, it biſects CD in L, and alſo the arch CD in A; DL is therefore equal to the ſine of the arch AC, and BH or LK being the ſine of AB, DK is the ſum of the ſines of the arches AC and AB, and CK the difference of their ſines: DB alſo is the ſum of the arches AC and AB, becauſe AD is equal to AC, and BC is their difference. Now, in the triangle DFC, becauſe FK is perpendicular to DC, (cor. prop. 1.) DK : KC : : tan. DFK : tan. CFK; but tan. DFK = tan. $\frac{1}{2}$ BD, becauſe the angle DFK (20. 3.) is the half of DEB, and is therefore meaſured by half the arch DB. For the ſame reaſon, tan. CFK = tan. $\frac{1}{2}$ BC; and conſequently, DK : KC : : tan. $\frac{1}{2}$ BD : tan. $\frac{1}{2}$ BC. But DK is the ſum of the ſines of AB and AC; KC the difference of their ſines; BD is the ſum of AB and AC, and BC their difference. Therefore, &c. Q. E. D.



COR. 1. Becauſe EL is the coſine of AC, and EH of AB, FK is the ſum of theſe coſines, and KB their difference. Now, FK : KB : : tan. FDK : tan. BDK; and tan. FDK = co-tan. DFK, becauſe DFK is the complement of FDK; therefore, FK : KB : : co-tan. DFK : tan. BDK;

that is, FK : KB : : co-tan. $\frac{1}{2}$ DB : tan. $\frac{1}{2}$ BC. The ſum of the coſines of two arches is therefore to the difference of the ſame coſines, as the co-tangent of half the ſum of the arches to the tangent of half their difference.

COR. 2. If the two arches AB and AC be together equal to 90°, the tangent of half their ſum, that is, of 45°, is equal to the radius. And the arch BC being the exceſs of DC above DB, or above 90°, the half of the arch BC will be equal to the exceſs of the half of DC above the half of DB, that is, to the exceſs of AC above 45°; therefore, when the ſum of two arches is 90°, the ſum of the ſines of thoſe arches is to their difference as the radius to the tangent of the difference between either of them and 45°.

PROP. IV.

THE ſum of the ſides of any plane triangle is to their difference, as the tangent of half the ſum of the angles at the baſe of the triangle, to the tangent of half their difference.

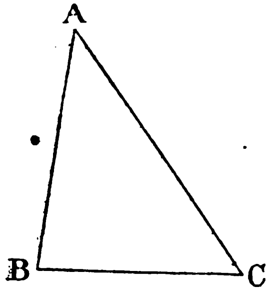
Let ABC be any plain triangle;

$$CA+AB : CA-AB :: \text{tan.}\,\frac{B+C}{2} : \text{tan.}\,\frac{B-C}{2}.$$

For (Prop. 2.) CA : AB :: ſin. B : ſin. C;
and therefore, by compoſition and diviſion,
CA+AB : CA—AB :: ſin. B + ſin. C : ſin. B — ſin. C.
But, by the laſt, ſin. B + ſin. C : ſin. B — ſin. C ::

$\text{tan.}\,\frac{B+C}{2} : \text{tan.}\,\frac{B-C}{2}$; therefore alſo,

$$CA+AB : CA-AB :: \text{tan.}\,\frac{B+C}{2} : \text{tan.}\,\frac{B-C}{2}. \quad \text{Q. E. D.}$$

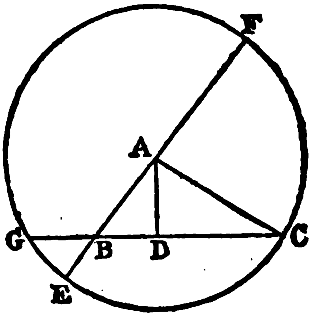


PROP. V.

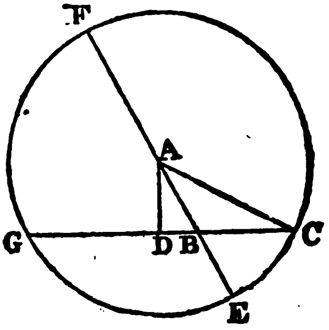
IN a plane triangle, if the perpendicular drawn to the baſe from the oppoſite angle fall within the triangle, the baſe is to the ſum of the ſides as the difference of the ſides to the difference of the ſegments of the baſe; but, if the perpendicular fall without the triangle, the baſe is to the ſum of the ſides as the difference of the ſides to the ſum of the ſegments of the baſe.

Let ABC be a plane triangle, and AD the perpendicular to the baſe from the oppoſite angle A.

1. Let AD be within the triangle. From A as a centre, with the diſtance AC, the greater of the two ſides, let a circle be deſcribed meeting AB produced in E and F, and CB in G. It is manifeſt, that FB is the ſum, and BE the difference of the ſides; and, ſince AD is drawn from the centre perpendicular to CG, CD is equal to DG, and therefore, BG is the difference of DC and DB, that is, of the ſegments of the baſe. Now, the rectangle CBG is equal to the rectangle FBE (35. 3.); and therefore, CB : BF : : BE : BG (16. 6.), that is, the baſe is to the ſum of the ſides, as the difference of the ſides to the difference of the ſegments of the baſe.



2. Let AD be without the triangle. It will be demonſtrated as in the laſt caſe, that CB : BF : : BE : BG; now, BG is equal to GD and DB, that is, to CD and DB, or to the ſum of the ſegments of the baſe; therefore, when the perpendicular falls without the triangle, the baſe is to the ſum of the ſides, as the difference of the ſides to the ſum of the ſegments of the baſe. Q. E. D.



Cor. The perpendicular AD falls within the triangle, when the ſquare of the greater ſide AC is leſs than the ſum of the ſquares on the other ſide, and on the baſe (13. 2.); and, when the contrary happens, the perpendicular falls without the triangle. Therefore, according as the ſquare on AC is leſs or greater than the ſquares on AB and BC, the triangle falls under the firſt or ſecond caſe of this propoſition.

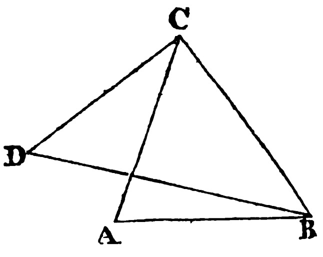
SCHOLIUM.

The foregoing propoſitions contain all that is neceſſary to the ſolution of the different queſtions in plane trigonometry. It is, however, uſeful in ſome of the caſes of oblique angled triangles, to be provided with other methods of calculation; on which account the following propoſitions are added.

PROP. VI.

IF, as the greater of any two ſides of a triangle to the leſs, ſo the radius to the tangent of a certain angle; then will the radius be to the tangent of the difference between that angle and half a right angle, as the tangent of half the ſum of the angles at the baſe of the triangle to the tangent of half their difference.

Let ABC be a triangle, the ſides of which are BC and CA, and the baſe AB, and let BC be greater than CA. Let DC be drawn at right angles to BC, and equal to AC; join BD, and becauſe (Prop. 1.) in the right angled triangle BCD, BC : CD : : R : tan. CBD, CBD is the angle, of which the tangent is to the radius as CD to BC, that is, as CA to BC, or as the leaſt of the two ſides of the triangle to the greateſt.



But BC+CD : BC—CD : : tan. $\frac{1}{2}$ (CDB+CBD) : tan. $\frac{1}{2}$ (CDB—CBD), (Prop. 5.);

and alſo, BC+CA : BC—CA :: tan. $\frac{1}{2}$ (CAB+CBA): tan. $\frac{1}{2}$ (CAB—CBA). Therefore, ſince CD=CA, tan. $\frac{1}{2}$ (CDB+CBD) : tan : $\frac{1}{2}$(CDB—CBD) :: tan. $\frac{1}{2}$ (CAB+CBA) : tan. $\frac{1}{2}$ (CAB—CBA). But, becauſe the angles CDB+CBD= 90°, tan. $\frac{1}{2}$ (CDB+CBD) : tan. $\frac{1}{2}$ (CDB—CBD) :: R : tan. (45°—CBD), (2. cor. Prop. 3.); therefore, R : tan. (45°—CBD) :: tan. $\frac{1}{2}$ (CAB+CBA) : tan. $\frac{1}{2}$ (CAB—CBA); and CBD was already ſhewn to be ſuch an angle that BC : CA :: R : tan. CBD. Therefore, &c. Q. E. D.

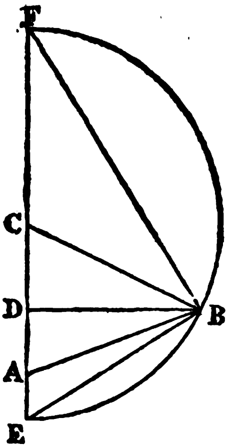
PROP. VII.

FOUR times the rectangle contained by any two ſides of a triangle, is to the rectangle contained by two ſtraight lines, of which one is the baſe or third ſide of the triangle increaſed by the difference of the two ſides, and the other the baſe diminiſhed by the difference of the ſame ſides, as the ſquare of the radius to the ſquare of the ſine of half the angle included between the two ſides of the triangle.

Let ABC be a triangle, of which AB is the baſe, and CB the greater of the two ſides. From B draw BD at right angles to AC.

Caſe 1. When ACB is an acute angle, ſo that BD falls within the triangle. From the centre C, with the diſtance CB, deſcribe the circle EBF, meeting AC produced in E and F. Join BE, BF; and it is evident, that EBF is a right angle, and alſo, that the angle EFB is half the angle ECB.

Then, (13. 2.) AC^2+CB^2, that is, $AC^2+CE^2=AB^2+2AC.CD$. But, (7. 2.) $AC^2+CE^2=2AC.CE+AE^2$, therefore, $2AC.CE+AE^2=AB^2+2AC.CD$; and if the rectangle $2AC.CD$ be taken from both, $2AC.CE-2AC.CD+AE^2=AB^2$, and therefore, becauſe $2AC.CE-2AC.CD=2AC.ED$, $2AC.ED+AE^2=AB^2$, or $2AC.ED=AB^2-AE^2$. But (5. 2.) $AB^2-AE^2=(AB+AE)(AB-AE)$, therefore, $2AC.ED=(AB+AE)(AB-AE)$.



Again, $AC:CE::2AC.DE:2CE.DE$, becauſe rectangles that have equal altitudes are as their baſes; and, for the ſame reaſon, $4AC.CE:4CE^2::2AC.DE:2CE.DE$, or alternately, $4AC.CE:2AC.DE::4CE^2:2CE.DE$. Now, $4CE^2=EF^2$, and $2CE.DE=FE.DE=EB^2$ (cor. 8. 6.), therefore, $4AC.CE:2AC.DE::EF^2:EB^2$. But

$EF:EB::R:\text{ſin. }EFB::R:\text{ſin. }\frac{1}{2}C$, whence,

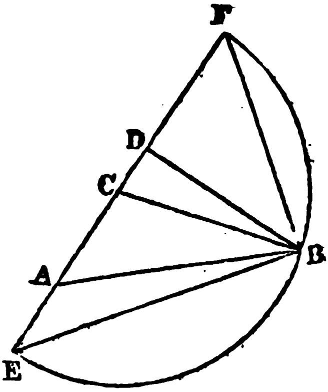
$EF^2:EB^2::R^2:(\text{ſin. }\frac{1}{2}C)^2$, and therefore,

$4AC.CE:2AC.DE::R^2:(\text{ſin. }\frac{1}{2}C)^2$ And it was proved, that $2AC.ED=(AB+AE)(AB-AE)$, therefore

$4AC.CE:(AB+AE)(AB-AE)::R^2:(\text{ſin. }\frac{1}{2}C)^2$.

Caſe 2. Let C be an obtuſe angle, ſo that the perpendicular DB falls without the triangle. Then, becauſe $AC^2+CB^2+2AC.CD=AB^2$ (12. 2.), or $AC^2+CE^2+2AC.CD=AB^2$; and becauſe $AC^2+CE^2=2AC.CE+AE^2$, $2AC.CE+2AC.CD+AE^2=AB^2$. Now, $2AC.CE+2AC.CD=2AC.ED$, therefore $2AC.ED+AE^2=AB^2$. Therefore, it will be demonſtrated as before, that

$$4AC.CE : (AB+AE)(AB-AE) :: R^2 : (\text{ſin.}\tfrac{1}{2}C)^2.$$



Now, $4AC.CE$, or $4AC.CB$ is four times the rectangle under the ſides, and AE being the difference of the ſides, $(AB+AE)(AB-AE)$ is the rectangle which has for one of its ſides the baſe increaſed by the difference of the two ſides of the triangle, and for its other ſide, the baſe diminiſhed by the ſame difference. Therefore, in any triangle, &c. Q. E. D.

Cor. Becauſe BDC is a right angle, $BC : CD :: R : \text{ſin.}CBD$; now, $\text{ſin.}\,CBD=\text{coſ.}\,C$, therefore, $BC : CD :: R : \text{coſ.}\,C$, and $2AC.CB : 2AC.CD :: R : \text{coſ.}\,C$. But, (13. 2.) $2AC.CD=AC^2+CB^2-AB^2$, therefore, $2AC.CB : AC^2+CB^2-AB^2 :: R : \text{coſ.}\,C$; that is, twice the rectangle AC.CB, is to the exceſs of the ſquares of AC and CB above the ſquare of AB, as radius to the co-ſine of the included angle ACB.

PROP.

PROP. VIII.

FOUR times the rectangle contained by any two ſides of a triangle, is to the rectangle contained by two ſtraight lines, of which one is the ſum of thoſe ſides increaſed by the baſe of the triangle, and the other the ſum of the ſame ſides diminiſhed by the baſe, as the ſquare of the radius to the ſquare of the co-ſine of half the angle included between the two ſides of the triangle.

The ſame conſtruction being made as in the laſt propoſition, it is plain, that AF is the ſum, and AE the difference of the ſides AC and CB, and alſo, that the angle FEB is half of the angle FCB. Now, if ACB be an acute angle AC^2+CB^2, or $AC^2+CF^2=AB^2+2AC.CD$; add $2AC.CF$ to both, and $AC^2+CF^2+2AC.CF$, that is, $AF^2=AB^2+2AC.CD+2AC.CF$; But $2AC.CD+2AC.CF=2AC.DF$, becauſe $CD+CF=DF$, and therefore, $AF^2=AB^2+2AC.DF$; and $2AC.DF=AF^2-AB^2=(AF+AB)(AF-AB)$, (2. 5.).

Again, $AC : CF :: 2AC.DF : 2CF.DF$, and $4AC.CF : 4CF^2 :: 2AC.DF : 2CF.DF$; and alternately, $4AC.CF : 2AC.DF :: 4CF^2 : 2CF.DF$; now, $4CF^2=EF^2$, and $2CF.DF=EF.DF=FB^2$, therefore, $4AC.CF : 2AC.DF :: EF^2 : FB^2$. Bnt $EF : FB :: R :$ ſin. FEB, and ſin. $FEB=$ coſ. EFB, becauſe EFB is the complement of FEB, ſo that

ſin. $FEB=$ coſ. $\frac{1}{2}ECB$; therefore, $EF : FB :: R :$ coſ. $\frac{1}{2}ECB$,

and $EF^2 : FB^2 :: R^2 : (\text{coſ.}\ \frac{1}{2}ECB.)^2$ Therefore,

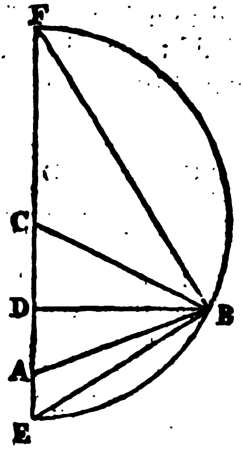
$4AC.CF : 2AC.DF :: R^2 : (\text{coſ.}\ \frac{1}{2}ECB)^2$, or

$4AC.CB : (AF+AB)(AF-AB) :: R^2 : (\text{coſ.}\ \frac{1}{2}ECB)^2$

In the ſame way, when the angle C is obtuſe, it will be demonſtrated, that $4AC.CB : (AF+AB)(AF-AB) :: R^2 :$

$(\text{coſ.}\ \frac{1}{2}ECB)^2$. Now, $4AC.CB$ is four times the rectangle under the ſides, and AF being the ſum of the ſides, $(AF+AB)(AF-AB)$ is the rectangle which has one of its ſides

s equal to the ſum of the ſides of the triangle increaſed by
baſe, and its other ſide equal to the ſum of the ſides of the
ngle diminiſhed by the baſe. Q. E. D.

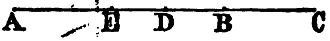


or. Becauſe $4AC.CB : (AF+AB)(AF-AB) :: R^2 :$
ſ. $\frac{1}{2}$ ECB)^2, and alſo (prop. 7.) $4AC.CB :$
B+AE)(AB—AE) : : R^2 : (ſin. $\frac{1}{2}$ ECB)^2; therefore,
equo, (AF+AB) (AF—AB) : (AB+AE)(AB—AE) : :
ſ. $\frac{1}{2}$ ECB)^2 : (ſin. $\frac{1}{2}$ ECB)^2.
, coſ. $\frac{1}{2}$ ECB : ſin. $\frac{1}{2}$ ECB : : R : tan. $\frac{1}{2}$ ECB;
refore, (AF+AB)(AF—AB) : (AB+AE)(AB—AE) : :
: (tan. $\frac{1}{2}$ ECB)^2; that is, the rectangle contained by two
ight lines, of which one is the ſum of the three ſides of a
ngle, and the other the ſum of two of the ſides of the tri-
le diminiſhed by the third, is to the rectangle contained by
ſtraight lines, of which one is the third ſide of the tri-
le increaſed by the difference of the other two, and the
er the third ſide diminiſhed by the difference of the other
, as the ſquare of the radius to the ſquare of the tangent
half the angle included between the two ſides of the tri-
le.

LEMMA II.

IF there be two unequal magnitudes, half their difference added to half their ſum is equal to the greater; and half their difference taken from half their ſum is equal to the leſs.

Let AB and BC be two unequal magnitudes, of which AB is the greater; ſuppoſe AC biſected in D, and AE equal to BC. It is manifeſt, that AC is the ſum, and EB the difference of the magnitudes. And becauſe AC is biſected in D, AD is equal to DC; but AE is alſo equal to BC, therefore DE is equal to DB, and DE or DB is half the difference of the magnitudes. But AB is equal to BD and DA, that is to half the difference added to half the ſum; and BC is equal to the exceſs of DC, half the ſum, above DB, half the difference. Therefore, &c. Q. E. D.



Cor. Hence, if the ſum and the difference of two magnitudes be given, the magnitudes themſelves may be found.

SOLUTION of the Caſes of Right Angled Triangles.

PROBLEM.

IN a right angled triangle of the three ſides and three angles, any two being given, beſides the right angle, and one of thoſe two being a ſide, it is required to find the other three.

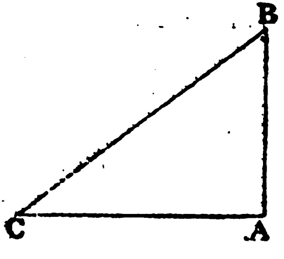
The reaſon of the limitation in this Prop. to thoſe caſes in which at leaſt one ſide is given, is manifeſt; for, if the angles only be given, the ſides cannot be found, but only their ratios, which are the ſame with the ratios of the ſines of the oppoſite angles.

It is evident, that when one of the acute angles of a right angled triangle is given, the other is given, being the complement of the former to a right angle; and alſo that the ſine of any of the acute angles is the co-ſine of the other.

As this general problem admits of ſeveral caſes, the ſolutions or rules for calculation, which all depend on the firſt Prop. may be conveniently exhibited in the form of a table.

When two ſides of a right angled triangle are given in numbers, the third may be found, either trigonometrically, as in the table, or by the 47th of the 1ſt of the Elem. Thus, becauſe $BC^2 = AB^2 + AC^2$, if AB and AC are given, BC is found, for it is the ſquare root of $AB^2 + AC^2$. In the ſame manner, if AB and BC are given, AC is found, being the ſquare root of $BC^2 - AB^2$.

GIVEN.	SOUGHT.	SOLUTION.	
CB and B, the hypotenuſe and an angle.	AC.	R : ſin. B : : CB : AC.	1
	AB.	R : coſ. B : : CB : AB.	1
AC and C; a ſide, and one of the acute angles.	BC.	Sin. B : R : : AC : BC.	2
	AB.	R : tan. C : : AC : AB.	3
CB and BA, the hypotenuſe and a ſide.	C.	CB : BA : : R : ſin. C.	4
	AC.	R : coſ. C : : CB : AC.	1
AC and AB; the two ſides.	C.	AC : AB : : R : tan. C.	5
	CB.	Coſ. C : R : : AC : CB.	2



SOLU-

SOLUTION of the Caſes of Oblique Angled Triangles.

PROBLEM.

IN any triangle of the three ſides and three angles, any three being given, and one of the three being a ſide, it is required to find the other three.

The reaſon of the reſtriction, that one of the things given muſt be a ſide, is evident; becauſe, if the angles only are given, the ſides of the triangle cannot be found, but only their ratios, which are the ſame with the ratios of the ſines of the oppoſite angles.

In this problem there are four caſes, in each of which the ſolution depends on ſome of the foregoing propoſitions.

CASE I.

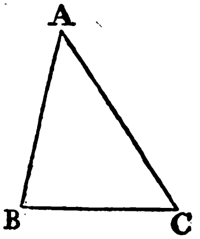
Two angles A and B, and one ſide AB, of a triangle ABC, being given, to find the other ſides.

SOLUTION.

Becauſe the angles A and B are given, C is alſo given, being the ſupplement of A+B; and, (Prop. 2.)

Sin. C : ſin. A : : AB : BC, alſo

Sin. C : ſin. B : : AB : AC.



CASE

CASE II.

Two ſides AB and AC, and the angle B oppoſite to one of them being given, to find the other angles A and C, and alſo the other ſide BC.

SOLUTION.

AC : AB : : ſin. B : ſin. C. Alſo, $A = 180° - B - C$; and ſin. B : ſin. A : : AC : CB, by caſe 1.

In this caſe, the angle C may have two values; for its ſine being found by the proportion above, the angle belonging to that ſine, may either be that which is found in the tables, or it may be the ſupplement of it, (Cor. def. 4.). But this ambiguity is removed when AC is greater than AB, for the angle ACB cannot in that caſe be obtuſe, (otherwiſe AB would be greater than AC), and therefore the greateſt of the two values of C is neceſſarily excluded. When AC is leſs than AB, C remains ambiguous, and there are two triangles that have the ſides AB, AC, and the angle B the ſame in both, but the angle oppoſite to AB in the one, is the ſupplement of the angle oppoſite to AB in the other.

CASE III.

Two ſides AB and AC, and the angle A between them being given, to find the other angles B and C, and alſo the ſide BC.

SOLUTION.

$$AB+AC : AB-AC :: \tan.\ \tfrac{1}{2}(B+C) : \tan.\ \tfrac{1}{2}(B-C.)$$

Hence, by the 2d Lemma, B and C are found.

Otherwiſe,

Find an angle E, ſuch that AB : BC : : R : tan. E ; then R : tan. (45°—E) : : tan. (B+C) : tan. (B—C). Whence, B and C are found, as before.

To find BC.

Having found B, ſin. B : ſin. A : : AC : BC.

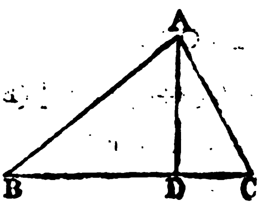
CASE IV.

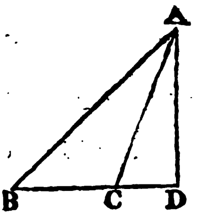
The three ſides AB, BC, AC, being given, to find the angles A, B, C.

SOLUTION

Take F ſuch that BC : BA+AC : BA—AC : F, then F is either the ſum or the difference of BD, DC, the ſegments of the baſe, (Prop. 5.) If F be greater than BC, F is the ſum, and BC the difference of BD, DC ; but, if F be leſs than BC, BC is the ſum, and F the difference of BD and DC. In either caſe, the ſum of BD and DC, and their difference being given, BD and DC are found.

Then, (prop. 1.) CA : CD : : R : coſ. C ;
and BA : BD : : R : coſ. B ; wherefore C and B are given, and conſequently A.





Otherwiſe,

Let D be the difference of the ſides AB, AC, then, (prop. 7.)

$$4AB.AC : (BC+D)(BC-D) :: R^2 : (\text{ſin. } \tfrac{1}{2}\, BAC)^2, \text{ or}$$

$$2\sqrt{AB.AC} : \sqrt{(BC+D)(BC-D)} :: R : \text{ſin. } \tfrac{1}{2}\, BAC.$$

Otherwiſe,

Let S be the ſum of the ſides BA and AC. Then (prop. 8.)

$$4AB.AC : (S+BC)(S-BC) :: R^2 : (\text{ſin. } \tfrac{1}{2}\, BAC)^2 \text{ or}$$

$$2\sqrt{AB.AC} : \sqrt{(S+BC)(S-BC)} :: R.\ \text{coſ. } \tfrac{1}{2}\, BAC.$$

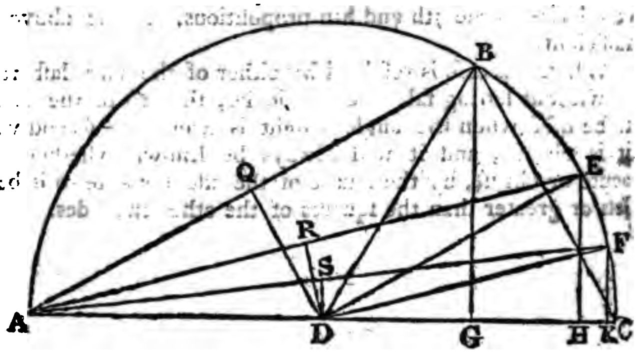
Other ſolutions of this caſe might alſo be given from the corollaries to the 7th and 8th propoſitions, but the above are ſufficient.

When this caſe is reſolved by either of the two laſt rules, or without letting fall a perpendicular, the firſt of the two is to be uſed when the angle ſought is acute; the ſecond when it is obtuſe; and it will always be known whether it is acute or obtuſe, by the ſquare of the ſide oppoſite to it being leſs or greater than the ſquares of the other two ſides.

SCHOLIUM.

In all the calculations performed by the preceding rules, tables of ſines and tangents are neceſſarily employed, the conſtruction of which it is therefore proper to explain. Theſe tables uſually contain the ſines, &c. to every minute of the quadrant from 1′ to 90°, and the firſt thing required to be done is to compute the ſine of 1′.

1. In the circle ABC the radius AD being unity, let BC be any arch, and CE its half; then, if DG the co-ſine of BC be known, the co-ſine of CE may be found. From E draw EH perpendicular to DC, ſo that EH may be the ſine, and DH the co-ſine of the arch CE. Join AB and BC, and let DQ be perpendicular to AB, then AB is biſected in Q. Now, the triangles DHE, AQD are equal, the angle DAQ being equal to the angle EDH, (becauſe each of them is the half of the angle CDB), and the angles DHE, AQD being right angles, and alſo the ſide AD equal to the ſide DE; therefore DH is equal to AQ. And becauſe ABC is a right angle, the rectangle CA, AG is equal to the ſquare of AB, (8. 6.) that



that is, to four times the ſquare of AQ or of DH; and therefore, half the rectangle DA, AH is equal to the ſquare of DH, that is, half the rectangle contained by the radius, and the ſum of the radius and the co-ſine of an arch, is equal to the ſquare of the co-ſine of half that arch. And hence, this arithmetical rule, to the co-ſine of any arch add the radius or 1, the ſquare root of half the ſum is the co-ſine of half the arch.

2. If therefore the ſtraight line CB be equal to the radius, the arch CB will be 60°, and DG will be equal to half the radius as was formerly ſhewn (8. 8.); and therefore DH the co-ſine of CE or of 30°, will be found by this propoſition. And if CE be biſected in F, if to DH 1 be added, and the ſum divided by 2, its ſquare root, will be equal to DK, the co-ſine of 15°. In the ſame manner is found the co-ſine of 7° 30′, of 3°, 45′, and ſo on till, after twelve biſections, the co-ſine of $52''. 44'''. 03''''. 45'''''$ is found. Now, from the co-ſine, the ſine of the ſame arch is found; for, from the ſquare of the radius, that is, from 1, take away the ſquare of the co-ſine, the remainder is the ſquare of the ſine, and therefore its ſquare root is the ſine itſelf. Thus, the ſine of $52''. 44'''. 03''''. 45'''''$ is found.

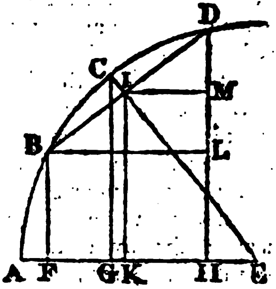
3. But it is manifeſt, that the ſines of very ſmall arches are to one another nearly as the arches themſelves. For it has been ſhewn, that the number of the ſides of an equilateral polygon inſcribed in a circle may be ſo great, that the perimeter of the polygon and the circumference of the circle may differ by a line leſs than any given line, or which is the ſame, may be nearly to one another in the ratio of equality. Therefore their like parts will alſo be nearly in the ratio of equality, ſo that the ſide of the polygon will be to the arch which it ſubtends nearly in the ratio of equality; and therefore, half the ſide of the polygon to half the arch ſubtended by it, that is to ſay, the ſine of any very ſmall arch will be to the arch itſelf, nearly in the ratio of equality. Therefore, if two arches are both very ſmall, the firſt will be to the ſecond as the ſine of the firſt to the ſine of the ſecond. Hence, from the ſine of $52''. 44'''. 03''''. 45'''''$ being found, the ſine of 1′ becomes known; for, as $52''. 44'''. 03''''. 45'''''$ to 1′, ſo the ſine of the former arch to the ſine of the latter. Thus, the ſine of 1′ is found =0.0002908882.

4. The ſine of 1′ being thus found, the ſines of 2′, of 3′ or of any number of minutes are found by the following propoſition.

THEOREM.

Let AB, AC, AD be three ſuch arches, that BC the difference of the firſt and ſecond is equal to CD the difference of the ſecond and third; the radius is to the co-ſine of the common difference BC as the ſine of AC the middle arch, to half the ſum of the ſines of AB and AD the extreme arches.

Draw CE to the centre; let BF, CG, and DH perpendicular to AE, be the ſines of the arches AB, AC, AD. Join BD, and let it meet CE in I; draw IK perpendicular to AE, alſo BL and IM perpendicular to DH. Then, becauſe the arch BD is biſected in C, EC will be at right angles to BD, and will biſect it in I, and BI will be the ſine, and EI the co-ſine of BC or CD. And, ſince BD is biſected in I and IM is parallel to BL, (2. 6.) LD is alſo biſected in M. Now BF is equal to HL, therefore, $BF+DH=DH+HL=DL+2LH=2LM+2LH=2MH$, or $2KI$; and therefore, IK is half the ſum of BF and DH. But becauſe the triangles CGE, IKE are equiangular, $CE : EI :: CG : IK$, and it has been ſhewn, that $EI =$ coſ. BC, and $IK = \frac{1}{2}(BF+DH)$; therefore $R :$ coſ. BC :: ſin. AC : $\frac{1}{2}$ (ſin. AB+ ſin. AD). Q. E. D.



Cor. Hence, if the point B coincides with A,

R : coſ. BC : : ſin. BC : $\frac{1}{2}$ ſin. BD, or the radius is to the co-ſine of the half of any arch, as the ſine of half the arch is to half the ſine of the whole arch.

Therefore, the ſine of 1′ being given, and alſo its co-ſine, becauſe by this Cor. R : coſ. 1′ : : ſin. 1′ : $\frac{1}{2}$ ſin. 2′; and ſo the ſine of 2′ is given. To find the ſine of 3′; by the preceding Theor. R : coſ. 1′ : : ſin. 2′ : $\frac{1}{2}$ (ſin. 1′+ſin. 3′); therefore, ſin. 1′+ ſin. 3′ is given, and ſin. 1′ being already known, ſin. 3′ is found.

In like manner, for the ſine of 4′,

R : coſ. 1′ : : ſin. 3′ : : $\frac{1}{2}$ (ſin. 2′+ſin. 4′); and ſo on for any number of minutes; and as the ratio of R to the co-ſine of 1′ remains always the ſame, the calculation by means of this Theorem is very eaſy.

It may be convenient to have the ſame rule otherwiſe expreſſed, let a, b, c, be three arches that differ by 1′, then,

R : coſ. 1′ : : ſin. b : $\frac{1}{2}$ (ſin. a+ſin. c), and if R = 1,

$\frac{1}{2}$ (ſin. a + ſin. c) = coſ. 1′ × ſin. b, and conſequently,
ſin. c = 2 coſ. 1′ × ſin. b — ſin. a.

In this manner, the table of ſines is computed. Then, becauſe the co-ſine of any arch is to the ſine as the radius to the tangent of the arch, the table of tangents may be calculated from this proportion. But it muſt be obſerved, that when the tangents of the arches under 45° are known, the tangents of thoſe above 45° may be more eaſily found by another rule.

For the tangents of the arches above 45° being the co-tangents of the arches under 45°, and the radius being a mean proportional between the the tangent and co-tangent of any arch, (1. Cor. def. 9.) if the difference between any arch and 45° be called d, tan. (45°—d) : 1 : : 1 : tan. (45°+d).

Laſtly, The ſecants are calculated by help of Cor. 2. Def. 9. where it is ſhewn, that the radius is a mean proportional beween the co-ſine and the ſecant of any arch. If therefore, a be any arch, coſ. a : 1 : : 1 : ſec. a.

5. The preceding Theorem is one of four, which, when arithmetically expreſſed, are frequently uſed in the application of trigonometry to the ſolution of the more difficult problems.

1mo, If in the laſt Theorem, the arch AC=A, BC=B, and EC=1, then AD=A+B, and AB=A—B; and by what has juſt been demonſtrated,

$$1 : \text{coſ. } B :: \text{ſin. } A : \tfrac{1}{2}\,\text{ſin.}\,(A+B) + \tfrac{1}{2}\,\text{ſin.}\,(A-B),$$

and therefore,

$$\text{ſin. } A \times \text{coſ. } B = \tfrac{1}{2}\,\text{ſin.}\,(A+B) + \tfrac{1}{2}\,\text{ſin.}\,(A-B).$$

2do, Becauſe BF IK, DH are parallel, the ſtraight lines BD and FH are cut proportionally, and therefore FH, the difference of the ſtraight lines FE and HE, is biſected in K; and therefore, as was ſhewn in the laſt Theorem, KE is half the ſum of FE and HE, that is, of the co-ſines of the arches AB and AD. But becauſe of the ſimilar triangles EGC, EKI, EC : EI :: GE : EK; now, GE is the co-ſine of AC, therefore,

$$R : \text{coſ. } BC :: \text{coſ. } AC : \tfrac{1}{2}\,\text{coſ. } AD + \tfrac{1}{2}\,\text{coſ. } AB,$$

$$\text{or } 1 : \text{coſ. } B :: \text{coſ. } A : \tfrac{1}{2}\,\text{coſ.}\,(A+B) + \tfrac{1}{2}\,\text{coſ.}\,(A-B);$$

and therefore,

$$\text{coſ. } A \times \text{coſ. } B = \tfrac{1}{2}\,\text{coſ.}\,(A+B) + \tfrac{1}{2}\,\text{coſ.}\,(A-B).$$

3tio, Again, the triangles IDM, CEG are equiangular, for the angles KIM, EID are equal, being each of them right angles, and therefore, taking away the angle EIM, the angle DIM is equal to the angle EIK, that is, to the angle ECG; and the angles DMI, CGE are alſo equal, being both right angles, and therefore, the triangles IDM, CGE have the ſides about their equal angles proportionals, and conſequently, EC : CG :: DI : IM; now, IM is half the difference of the co-ſines FE and EH, therefore

$$R : \text{ſin. } AC : \text{ſin. } BC : \tfrac{1}{2}\,\text{coſ. } AB - \tfrac{1}{2}\,\text{coſ. } AD,$$

$$\text{or } 1 : \text{ſin. } A : \text{ſin. } B :: \tfrac{1}{2}\,\text{coſ.}\,(A-B) - \tfrac{1}{2}\,\text{coſ.}\,(A+B);$$

and alſo,

$$\text{ſin. } A \times \text{ſin. } B = \tfrac{1}{2}\,\text{coſ.}\,(A-B) - \tfrac{1}{2}\,\text{coſ.}\,(A+B).$$

4to, Laſtly, In the ſame triangles ECG, DIM, EC : EG :: ID : DM; now, DM is half the difference of the ſines DH and BF, therefore,

$$\text{R} : \text{coſ. AC} :: \text{ſin. BC} : \tfrac{1}{2}\,\text{ſin. AD} - \tfrac{1}{2}\,\text{ſin. AB}.$$

$$\text{or } 1 : \text{coſ. A} :: \text{ſin. B} : \tfrac{1}{2}\,\text{ſin.}\,(\text{A}+\text{B}) - \tfrac{1}{2}\,\text{ſin.}\,(\text{A}-\text{B});$$

and therefore,

$$\text{coſ. A} \times \text{ſin. B} = \tfrac{1}{2}\,\text{ſin.}\,(\text{A}+\text{B}) - \tfrac{1}{2}\,\text{ſin.}\,(\text{A}-\text{B}).$$

6. If therefore A and B be any two arches whatſoever, the radius being ſuppoſed 1;

$$\text{I. ſin. A} \times \text{coſ. B} = \tfrac{1}{2}\,\text{ſin.}\,(\text{A}+\text{B}) + \tfrac{1}{2}\,\text{ſin.}\,(\text{A}-\text{B}).$$

$$\text{II. coſ. A} \times \text{coſ. B} = \tfrac{1}{2}\,\text{coſ.}\,(\text{A}-\text{B}) + \tfrac{1}{2}\,\text{coſ.}\,(\text{A}+\text{B}).$$

$$\text{III. ſin. A} \times \text{ſin. B} = \tfrac{1}{2}\,\text{coſ.}\,(\text{A}-\text{B}) - \tfrac{1}{2}\,\text{coſ.}\,(\text{A}+\text{B}).$$

$$\text{IV. coſ. A} \times \text{ſin. B} = \tfrac{1}{2}\,\text{ſin.}\,(\text{A}+\text{B}) - \tfrac{1}{2}\,\text{ſin.}\,(\text{A}-\text{B}).$$

From theſe four Theorems are alſo deduced other four.

For, by adding the firſt and fourth together,

$$\text{ſin. A} \times \text{coſ. B} + \text{coſ. A} \times \text{ſin. B} = \text{ſin.}\,(\text{A}+\text{B}).$$

Alſo, by taking the fourth from the firſt,

$$\text{ſin. A} \times \text{coſ. B} - \text{coſ. A} \times \text{ſin. B} = \text{ſin.}\,(\text{A}-\text{B}).$$

Again, adding the ſecond and third,

$$\text{coſ. A} \times \text{coſ. B} + \text{ſin. A} \times \text{ſin. B} = \text{coſ.}\,(\text{A}-\text{B});$$

And, laſtly, ſubtracting the third from the ſecond,

$$\text{coſ. A} \times \text{coſ. B} - \text{ſin. A} \times \text{ſin. B} = \text{coſ.}\,(\text{A}+\text{B}).$$

ELEMENTS

OF

SPHERICAL TRIGONOMETRY

DEFINITIONS.

I.

A Great circle of the ſphere is any circle on the ſuperficies of the ſphere of which the plane paſſes through the centre of the ſphere, and of which the centre therefore is the ſame with the centre of the ſphere.

II.

The pole of a great circle of the ſphere is a point in the ſuperficies of the ſphere, from which all ſtraight lines drawn to the circumference of the circle are equal.

III.

A ſpherical angle is that which on the ſuperficies of a ſphere is contained by two arches of great circles, and is the ſame with the inclination of the planes of theſe great circles.

IV.

A ſpherical triangle is a figure upon the ſuperficies of a ſphere comprehended by three arches of three great circles, each of which is leſs than a ſemicircle.

PROP.

PROP. I.

ANY two great circles of a ſphere biſect one another.

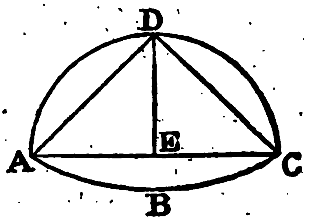
For, as they have the ſame centre, their common ſection is a diameter of both, and therefore biſects both.

PROP. II.

THE arch of a great circle between the pole and the circumference of another great circle is a quadrant.

Let ABC be a great circle, and D its pole; if a great circle CD paſs through D, and meet ABC in C, the arch DC is a quadrant.

Let the great circle CD meet ABC again in A, and let AC be the common ſection of the great circles, which will paſs through E the centre of the ſphere: Join DE, DA, DC; By def. 2. DA, DC are equal, and AE, EC are alſo equal, and DE is common; therefore (8. 1.) the angles DEA, DEC are equal; wherefore the arches DA, DC are equal, and conſequently each of them is a quadrant. Q. E. D.



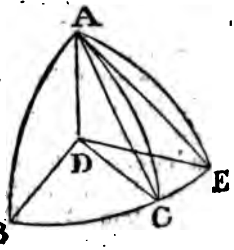
COR. The circle ABC has two poles, one on each ſide of its plane, which are the extremities of a diameter of the ſphere perpendicular to the plane ABC; and no other point but theſe two can be a pole of the circle ABC.

PROP.

PROP. III.

IF the pole of a great circle be the ſame with the interſection of other two great circles; the arch of the firſt mentioned circle intercepted between the other two, is the meaſure of the ſpherical angle which the ſame two circles make with one another.

Let the great circles BA, CA on the ſuperficies of a ſphere, of which the centre is D, interſect one another in A, and let BC be an arch of another great circle, of which the pole is A; BC is the meaſure of the ſpherical angle BAC.



Join AD, DB, DC; ſince A is the pole of BC, AB, AC are quadrants, and the angles ADB, ADC are right angles; therefore (4. def. 7.) the angle CDB is the inclination of the planes of the circles AB, AC, and is (def. 3. Sp. T.) equal to the ſpherical angle BAC. Q. E. D.

Cor. 1. If through the point A, two quadrants AB, AC, be drawn, the point A will be the pole of the great circle BC, paſſing through their extremities B, C.

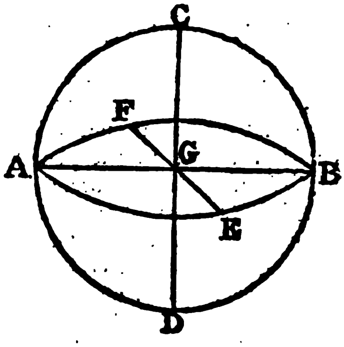
Join AC, and draw AE, a ſtraight line to any other point E in BC; join DE: Since AC, AB are quadrants, the angles ADB, ADC are right angles, and AD will be perpendicular to the plane of BC: Therefore the angle ADE is a right angle, and AD, DC are equal to AD, DE each to each; therefore AE, AC are equal, and A is the pole of BC, by def. 2.

PROP. IV.

IF there be two great circles of a ſphere, of which the firſt paſſes through the poles of the ſecond; the ſecond alſo paſſes through the poles of the firſt.

Let ACBD and AEBF be two circles, the one of which ACBD paſſes through C and D, the poles of the other AEBF; the circle AEBF paſſes through the poles of the circle ACBD.

Let G be the centre of the ſphere; join CG, GD, which will be in the ſame ſtraight line: Draw AGB the diameter, which is the common ſection of the circles ACBD, AEBF; and in the plane of the circle AEBF draw from the point G the ſtraight line EGF perpendicular to AB.



Then, becauſe C and D are the poles of the circle AEBF, the arch intercepted between either of them and any point in the circumference AEBF, is a quadrant (Prop. 3.), and therefore the ſtraight line CD is perpendicular to the plane of the circle AEBF; wherefore alſo the plane of the circle ACBD, which paſſes through CD, is perpendicular (17. 7.) to the plane AEBF. And ſince GE in the plane AEBF, is perpendicular to AB, the common ſection of the two planes, it is perpendicular to the plane (2. def. 7.) ACBD. The arch of a great circle, therefore, intercepted between the point E and any point of the circumference ACBD is a quadrant, and therefore, (Prop. 3.) the point E is the pole of the circle ACBD. For the ſame reaſon, F is the other pole of the circle ACBD; and E and F are in the circumference of the circle AEBF. Q. E. D.

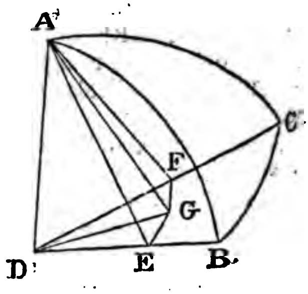
PROP. V.

IN iſoſceles ſpherical triangles the angles at the baſe are equal.

Let ABC be a ſpherical triangle, having the ſide AB equal to the ſide AC; the ſpherical angles ABC and ACB are equal.

Let D be the centre of the ſphere; join DB, DC, DA, and from A on the ſtraight lines DB, DC, draw the perpendiculars AE, AF; and from the points E and F draw in the plane DBC the ſtraight lines EG, FG perpendicular to DB and DC, meeting one another in G: Join AG.

Becauſe DE is at right angles to each of the ſtraight lines AE, EG, it is at right angles to the plane AEG, which paſſes through AE, EG, (4. 7.); and therefore every plane that paſſes through DE is at right angles to the plane AEG (17. 7.); wherefore, the plane DBC is at right angles to the plane AEG. For the ſame reaſon, the plane DBC is at right angles to the plane AFG, and therefore AG, the common ſection of the planes AFG, AEG is at right angles (18. 7.) to the plane DBC, and the angles AGE, AGF are conſequently right angles.



But, ſince the arch AB is equal to the arch AC, the angle ADB is equal to the angle ADC. Therefore the triangles ADE, ADF, have the angles EDA, FDA equal, as alſo the angles AED, AFD, which are right angles; and they have the ſide AD common, therefore the other ſides are equal, viz. AE to AF, (26. 1.) and DE to DF. Again, becauſe the angles AGE, AGF are right angles, the ſquares on AG and GE are equal to the ſquare of AE; and the ſquares of AG and GF to the ſquare of AF. But the ſquares of AE and AF are equal, therefore the ſquares of AG and GE are equal to the ſquares of AG and GF, and taking away the common ſquare

e of AG, the remaining ſquares of GE and GF are equal,
GE is therefore equal to GF. Wherefore, in the triangles
G, AEG, the ſide GF is equal to the ſide GE, and AF
een proved to be equal to AE, and the baſe AG is com-
, therefore the angle AFG is equal to the angle AEG
.). But the angle AFG is the angle which the plane
C makes with the plane DBC (4. def. 7.), becauſe FA
FG, which are drawn in theſe planes, are at right angles
F, the common ſection of the planes. The angle AFG
ef. Sp. T.) is therefore equal to the ſpherical angle ACB;
or the ſame reaſon, the angle AEG is equal to the ſphe-
angle ABC. But the angles AFG, AEG are equal.
efore the ſpherical angles ACB, ABC are alſo equal.
. D.

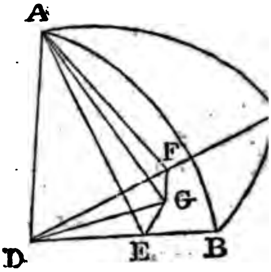
PROP. VI.

the angles at the baſe of a ſpherical triangle be
qual, the triangle is iſoſceles.

t ABC be a ſpherical triangle having the angles ABC,
B equal to one another; the ſides AC and AB are alſo
.

t D be the centre of the ſphere; join DB, DC, DA,
rom A on the ſtraight lines DB, DC, draw the perpen-
ars AE, AF; and from the points E and F, draw in
plane DBC the ſtraight lines EG, FG perpendicular
B and DC, meeting one another in G; join AG.
en, it may be proved as was done in the laſt Prop. that
is at right angles to the plane BCD, and that therefore the
s AGF, AGE are right angles, and alſo that the angles
, AEG are equal to the angles which the planes DAC,
B make with the plane DBC. But becauſe the ſpherical
s ACB, ABC are equal, th   es which the planes
, DAB make with the pl   are equal, (3. def.
Tr.) and therefore the   G, AEG are alſo
. The triangl   herefore two angles
e one equal to   r, and they have
he ſide AG   are equal, and the
AF is equal

Again,

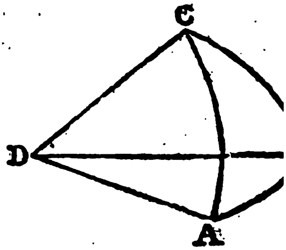


Again, becauſe the triangles ADF, ADE are right an at F and E, the ſquares of DF and FA are equal to ſquare of DA, that is, to the ſquares of DE and EA; now, the ſquare of AF is equal to the ſquare of AE, therefore the ſquare of DF is equal to the ſquare of DE, and the ſide DF to the ſide DE. Therefore in the triangles DAF, DAE, becauſe DF is equal to DE, and DA common, and alſo AF equal to AE, the angle ADF is equal to the angle ADE; therefore alſo, the arches AC and which are the meaſures of the angles ADF and ADE equal to one another; and the triangle ABC is iſoſc Q. E. D.

PROP. VII.

ANY two ſides of a ſpherical triangle are gr er than the third.

Let ABC be a ſpherical triangle, any two ſides AB, are greater than the third ſide AC.



Let D be the centre of the ſphere; join DA, DB, DC.

The ſolid angle at D is contained by three plane angles ADB, ADC, BDC; (19. 7.) any two of which ADB, BDC are greater than the third ADC, that is, any two ſides AB, BC of the ſpherical triangle ABC, are greater than the third Q. E. D.

PROP. VIII.

THE three ſides of a ſpherical triangle are leſs than a circle.

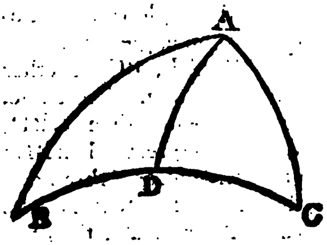
Let ABC be a ſpherical triangle as before, the three ſides AB, BC, AC are leſs than a circle.

Let D be the centre of the ſphere: The ſolid angle at D is contained by three plain angles BDA, BDC, ADC, which together are leſs than four right angles, (20. 7.) therefore the ſides AB, BC, AC together, will be leſs than four quadrants, that is leſs than a circle. Q. E. D.

PROP. IX.

IN a ſpherical triangle the greater angle is oppoſite to the greater ſide; and converſely.

Let ABC be a ſpherical triangle, the greater angle A is oppoſed to the greater ſide BC.



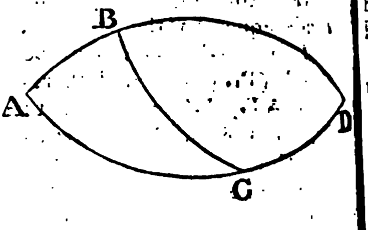
Let the angle BAD be made equal to the angle B, and then BD, DA will be equal, (Prop. 6.) and therefore AD, DC are equal to BC; but AD, DC are greater than AC, (Prop. 7.), therefore BC is greater than AC, that is, the greater angle A is oppoſite to the greater ſide BC. The converſe is demonſtrated as Prop. 19. 1. Elem. Q. E. D.

PROP. X.

IN a ſpherical triangle, according as the ſum of two of the ſides is greater than a ſemicircle, equal to it, or leſs, the interior angle at the baſe is greater than the exterior and oppoſite angle at the baſe, equal to it, or leſs; and the ſum of the two interior angles at the baſe greater than two right angles, equal to two right angles, or leſs than two right angles.

Let ABC be a ſpherical triangle, of which the ſides are AB and BC; produce the ſide AB and the baſe AC till they meet again in D; then, the arch ABD is a ſemicircle, and the ſpherical angles at A and D are equal, becauſe each of them is the inclination of the circle ABD to the circle ACD.

1. If AB, BC be equal to a ſemicircle, that is, to AD, BC will be equal to BD, and therefore (Prop. 5.) the angle D, or the angle A will be equal to the angle BCD.



2, If AB, BC together be greater than a ſemicircle, that is, greater than ABD, BC will be greater than BD; and therefore (Prop. 9.), the angle D, that is, the angle A, is greater than the angle BCD.

3. In the ſame manner it is ſhewn, if AB, BC together be leſs than a ſemicircle, that the angle A is leſs than the angle BCD. And ſince the angles BCD, BCA are equal to two right angles, if the angle A be greater than BCD, A and ACB together will be greater than two right angles. If A be equal to BCD, A and ACB together will be equal to two

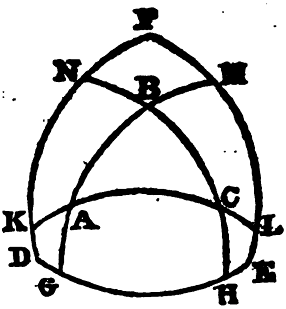
right angles; and if A be leſs than BCD, A and ACB will be leſs than two right angles. Q. E. D.

PROP. XI.

IF the angular points of a ſpherical triangle be made the poles of three great circles, theſe three circles by their interſections will form a triangle which is ſaid to be ſupplemental to the former; and the two triangles are ſuch, that the ſides of the one are the ſupplements of the arches which meaſure the angles of the other.

Let ABC be a ſpherical triangle; and from the points A, B, and C as poles, let the great circles FE, ED, DF be deſcribed, interſecting one another in F, D and E; the ſides of the triangle FED are the ſupplements of the meaſures of the angles A, B, C, viz. FE of the angle BAC, DE of the angle ABC, and DF of the angle ACB: And again, AC is the ſupplement of the angle DFE, AB of the angle FED, and BC of the angle EDF.

Let AB produced meet DE, EF in G, M; let AC meet FD, FE in K, L; and let BC meet FD, DE in N, H.



Since A is the pole of FE, and the circle AC paſſes through A, EF will paſs through the pole of AC (Prop. 4.), and ſince AC paſſes through C, the pole of FD, FD will paſs through the pole of AC; therefore the pole of AC is in the point F, in which the arches DF, EF interſect each other. In the ſame manner, D is the pole of BC, and E the pole of AB.

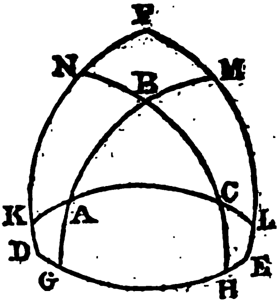
And ſince F, E are the poles of AL, AM, the arches FL and EM are quadrants, and FL, EM together, that is, FE and ML together are equal to a ſemicircle. But ſince A is the pole of ML, ML is the meaſure of the angle BAC, (Prop. 3.) conſequently FE is the ſupplement of the meaſure of the angle BAC. In the ſame manner, ED, DF are the ſupplements of the meaſures of the angles ABC, BCA.

Since likewiſe CN, BH are quadrants, CN, BH together, that is, NH, BC together, are equal to a ſemicircle; and ſince D is the pole of NH, NH is the meaſure of the angle FDE, therefore the meaſure of the angle FDE is the ſupplement of the ſide BC. In the ſame manner, it is ſhewn that the meaſures of the angles DEF, EFD are the ſupplements of the ſides AB, AC, in the triangle ABC. Q. E. D.

PROP. XII.

THE three angles of a ſpherical triangle are greater than two right angles, and leſs than ſix right angles.

The meaſures of the angles A, B, C, in the triangle ABC, together with the three ſides of the ſupplemental triangle DEF, are (Prop 11.) equal to three ſemicircles; but the three ſides of the triangle FDE, are (Prop. 8.) leſs than two ſemicircles; therefore the meaſures of the angles A, B, C are greater than a ſemicircle; and hence the angles A, B, C are greater than two right angles.



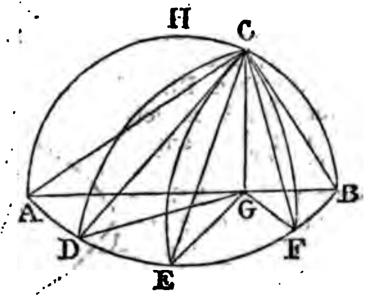
And becauſe all the external and internal angles of any triangle are equal to ſix right angles; therefore, all the internal angles are leſs than ſix right angles.

PROP. XIII.

IF to the circumference of a great circle, from a point which is not the pole of it, arches of great circles be drawn; the greateſt of theſe arches is that which paſſes through the pole of the firſt mentioned circle, and the ſupplement of it is the leaſt; and of the others, that which is nearer to the greateſt is greater than that which is more remote.

Let ADB be the circumference of a great circle, of which the pole is H, and let C be any other point; through C and H let the ſemicircle ACB be drawn meeting the circle ADB in A and B; and let the arches CD, CE, CF alſo be deſcribed. From C draw CG perpendicular to AB, and then, becauſe the circle AHCB which paſſes through H, the pole of the circle ADB, is at right angles to ADB, CG is perpendicular to the plane ADB. Join GD, GE, GF, CD, CE, CF, CA, CB.

Of all the ſtraight lines drawn from G to the circumference ADB, GA is the greateſt, and GB the leaſt (7. 3.); and GD which is nearer to GA is greater than GE, which is more remote. But the triangles CGA CGD are right angled at G, and they have the common ſide CG; therefore the ſquares of CG, GA together, that is, the ſquare of CA, is greater than the ſquares of CG, GD together, that is, than the ſquare of CD; therefore CA is greater than CD, and the arch CA than the arch CD. In the ſame manner, ſince GD is greater than GE, and GE than GF, it is ſhewn that CD is greater than CE, and CE than CF, and conſequently, the arch CD greater than the arch CE, and the arch CE greater than the arch CF. Alſo, becauſe AG is the greateſt, and GB the leaſt of all the lines drawn from G, CA is the greateſt, and CB the leaſt of all the lines drawn from C, and therefore the arch CA is the greateſt, and CB, its ſupplement, the leaſt of all the arches drawn through C. Q. E. D.



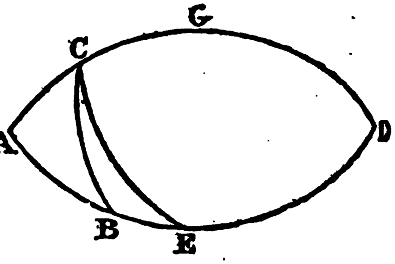
PROP.

PROP. XIV.

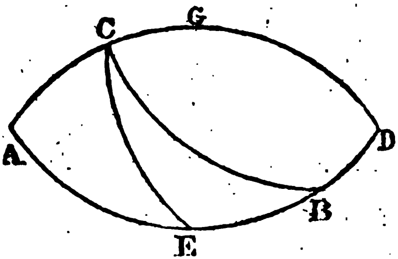
IN a right angled ſpherical triangle the ſides are of the ſame affection with the oppoſite angles, that is, if the ſides be greater or leſs than quadrants, the oppoſite angles will be greater or leſs than right angles.

Let ABC be a ſpherical triangle right angled at A, any ſide AB, will be of the ſame affection with the oppoſite angle ACB.

Caſe 1. Let AB be leſs than a quadrant. Let AE be a quadrant, and EC an arch of a great circle paſſing through E, C. Since A is a right angle, and AE a quadrant, E is the pole of the great circle AC, and ECA a right angle; but ECA is greater than BCA, therefore BCA is leſs than a right angle. Q. E. D.



Caſe 2. Let AB be greater than a quadrant, make AE equal to a quadrant, and let a great circle paſs through C, E. ECA is a right angle as before, and BCA is greater than ECA, that is, greater than a right angle. Q. E. D.



PROP. XV.

IF the two ſides of a right angled ſpherical triangle be of the ſame affection, the hypotenuſe will be leſs than a quadrant; and if they be of different affection, the hypotenuſe will be greater than a quadrant.

Let ABC be a right angled ſpherical triangle, if the two ſides AB, AC be of the ſame or of different affection, the hypotenuſe BC will be leſs or greater than a quadrant.

Caſe 1. Let AB, AC be each leſs than a quadrant. Let AE, AG be quadrants; G will be the pole of AB; and E the pole of AC, and EC a quadrant; but, (Prop. 13.) CE is greater than CB, ſince CB is farther off from CGD than CE. In the ſame manner, it is ſhewn that CB, in the triangle CBD, where the two ſides CD, BD are each greater than a quadrant, is leſs than CE, that is, leſs than a quadrant.

Caſe 2. Let AC be leſs, and AB greater than a quadrant; then the hypotenuſe BC will be greater than a quadrant; for, let AE be a quadrant, then E is the pole of AC, and EC will be a quadrant. But CB is greater than CE (Prop. 13.), ſince AC paſſes through the pole of ABD. Q. E. D.

Cor. 1. Hence, converſely, if the hypotenuſe of a right angled triangle be greater or leſs than a quadrant, the ſides will be of different or the ſame affection.

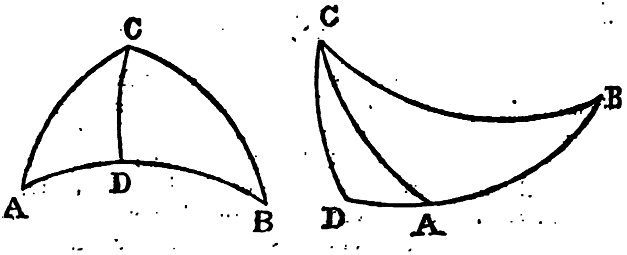
Cor. 2. Since (Prop. 14.) the angles of a right angled ſpherical triangle have the ſame affection with the oppoſite ſides, therefore, according as the hypotenuſe is greater or leſs than a quadrant, the angles will be of different or of the ſame affection.

PROP. XVI.

IN any ſpherical triangle, if the perpendicular upon the baſe from the oppoſite angle fall within the triangle, the angles at the baſe are of the ſame affection; and if the perpendicular fall without the triangle, the angles at the baſe are of different affection.

Let ABC be a ſpherical triangle; and let the arch CD be drawn from C perpendicular to the baſe AB.

1. Let CD fall within the triangle; then ſince ADC, BDC are right angled ſpherical triangles, the angles A, B muſt each be of the ſame affection with CD (Prop. 14.).



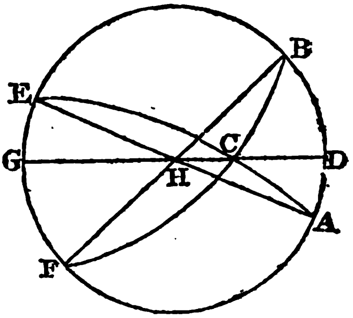
2. Let CD fall without the triangle; then (Prop. 14.) the angle B is of the ſame affection with CD; and the angle CAD is of the ſame affection with CD; therefore the angles CAD and B are of the ſame affection, and the angles CAB and B of different affections.

Cor. Hence, if the angles A and B be of the ſame affection, the perpendicular will fall within the baſe; for, if it did not, A and B would be of different affection. And, if the angles A and B be of oppoſite affection, the perpendicular will fall without the triangle; for, if it did not, the angles A and B would be of the ſame affection, contrary to the ſuppoſition.

PROP. XVII.

IF to the baſe of a ſpherical triangle a perpendicular be drawn from the oppoſite angle, which either falls within the triangle, or is the neareſt of the two that fall without; the leaſt of the ſegments of the baſe is adjacent to the leaſt of the ſides of of the triangle, or to the greateſt, according as the ſum of the ſides is leſs or greater than a ſemicircle.

Let ABEF be a great circle of a ſphere, H its pole, and GHD any circle paſſing through H, which therefore is perpendicular to the circle ABEF. Let A and B be two points in the circle ABEF on oppoſite ſides of the point D, and let D be nearer to A than to B, and let C be any point in the circle GND, between H and D. Through the points A and C, B and C, let the arches AC and BC be drawn, and let them be produced till they meet the circle ABEF in the points E and F, then the arches ACE, BCF are ſemicircles. Alſo ACB, ACF, CFE, ECB are four ſpherical triangles contained by arches of the ſame circles, and having the ſame perpendiculars CD and CG.



1. Now, becauſe CE is nearer to the arch CHG than CB is, CE is greater than CB, and therefore CE and CA are greater than CB and CA, wherefore CB and CA are leſs than a ſemicircle; but becauſe AD is by ſuppoſition leſs than DB, AC is alſo leſs than CB, (Prop. 13.), and therefore in this caſe, viz. when the perpendicular falls within the triangle, and when the ſum of the ſides is leſs than a ſemicircle, the leaſt ſegment is adjacent to the leaſt ſide.

2. Again, in the triangle FCA the two ſides FC and CA are leſs than a ſemicircle; for, ſince AC is leſs than CB, AC and CF are leſs than BC and CF. Alſo, AC is leſs than CF, becauſe it is more remote from CHG than CF is; therefore the leaſt ſegment of the baſe AD is in this caſe alſo adjacent to the leaſt ſide.

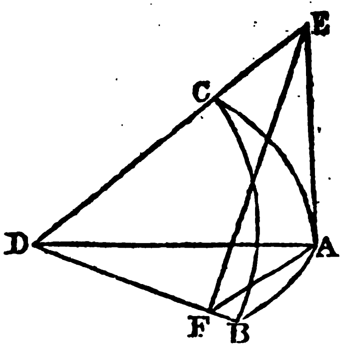
3. But in the triangle FCE the two ſides FC and CE are greater than a ſemicircle; for, ſince FC is greater than CA, FC and CE are greater than AC and CE. And becauſe AC is leſs than CB, EC is greater than CF, and EC is therefore nearer to the perpendicular CHG than CF is, wherefore EG is the leaſt ſegment of the baſe, and is adjacent to the greater ſide.

4. In the triangle ECB the two ſides EC, CB are greater than a ſemicircle; for, ſince by ſuppoſition CB is greater than CA, EC and CB are greater than EC and CA. Alſo, EC is greater than CB, wherefore in this caſe, alſo, the leaſt ſegment of the baſe EG is adjacent to the greateſt ſide of the triangle. Therefore, &c. Q. E. D.

PROP. XVIII.

IN right angled ſpherical triangles, the ſine of either of the ſides about the right angle, is to the radius of the ſphere, as the tangent of the remaining ſide is to the tangent of the angle oppoſite to that ſide.

Let ABC be a triangle, having the right angle at A; and let AB be either of the ſides, the ſine of the ſide AB will be to the radius, as the tangent of the other ſide AC to the tangent of the angle ABC, oppoſite to AC. Let D be the centre of the ſphere; join AD, BD, CD, and let AF be drawn perpendicular to BD, which therefore will be the ſine of the arch AB, and from the point F, let there be drawn in the plane BDC the ſtraight line FE at right angles to BD, meeting DC in E, and let AE be joined. Since therefore the ſtraight line DF is at right angles to both FA and FE, it will alſo be at right angles to the plane AEF (4. 7.), wherefore the plane ABD, which paſſes through DF is perpendicular to the plane AEF (17. 7.), and the plane AEF perpendicular to ABD: But the plane ACD or AED is alſo perpendicular to the ſame ABD: Therefore the com-



mon ſection, viz. the ſtraight line AE, is at right angles to the plane ABD, (18. 7.), and EAF, EAD are right angles. Therefore AE is the tangent of the arch AC; and in the rectilineal triangle AEF, having a right angle at A, AF is to the radius as AE to the tangent of the angle AFE, (1. Pl. Tr.); but AF is the ſine of the arch AB, and AE the tangent of the arch AC, and the angle AFE is the inclination of the planes CBD, ABD, (4. def. 7.) or the ſpherical angle ABC: Therefore the ſine of the arch AB is to the radius as the tangent of the arch AC to the tangent of the oppoſite angle ABC. Q. E. D.

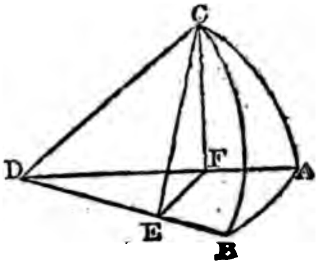
Cor. And ſince by this propoſition the ſine of the ſide AB is to the radius, as the tangent of the other ſide AC to the tangent of the angle ABC oppoſite to that ſide; and as the radius is to the co-tangent of the angle ABC, ſo is the tangent of the ſame angle ABC to the radius, (2. Cor. def. 9. Pl. Tr.) by equality, the ſine of the ſide AB is to the co-tangent of the angle ABC adjacent to it, as the tangent of the other ſide AC to the radius.

PROP. XIX.

IN right angled ſpherical triangles the ſine of the hypotenuſe is to the radius, as the ſine of either ſide is to the ſine of the angle oppoſite to that ſide.

Let the triangle ABC be right angled at A, and let AC be either of the ſides; the ſine of the hypotenuſe BC will be to the radius as the ſine of the arch AC is to the ſine of the angle ABC.

Let D be the centre of the ſphere, and let CE be drawn perpendicular to DB, which will therefore be the ſine of the hypotenuſe BC; and from the point E let there be drawn in the plane ABD the ſtraight line EF perpendicular to DB and let CF be joined: CF will be at right angles to the plane ABD, as was ſhown in the preceding propoſition of the ſtraight line EA: Wherefore CFD, CFE are right angles, and CF is the ſine of the arch AC; and in the triangle CFE, having the right angle



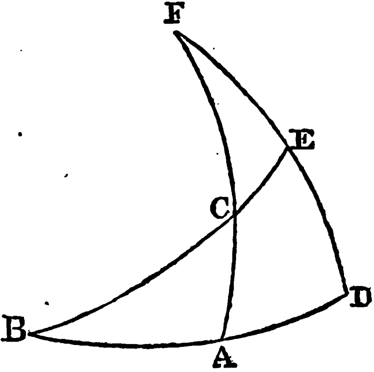
angle CFE, CE is to the radius, as CF to the ſine of the angle CEF, (1. Pl. Tr.). But, ſince CE, FE are at right angles to DEB, which is the common ſection of the planes CBD, ABD, the angle CEF is equal to the inclination of theſe planes, (4. def. 7.), that is, to the ſpherical angle ABC. The ſine, therefore, of the hypotenuſe CB is to the radius as the ſine of the ſide AC is to the ſine of the oppoſite angle ABC. Q. E. D.

Cor. Of theſe three, viz. the hypotenuſe, a ſide, and the angle oppoſite to that ſide, any two being given, the third may be found.

PROP. XX.

In right angled ſpherical triangles, the co-ſine of the hypotenuſe is to the radius as the co-tangent of either of the angles is to the tangent of the remaining angle.

Let ABC be a ſpherical triangle, having a right angle at A, the co-ſine of the hypotenuſe BC will be to the radius as the co-tangent of the angle ABC to the tangent of the angle ACB.



Deſcribe the circle DE, of which B is the pole, and let it meet AC in F, and the circle BC in E; and ſince the circle BD paſſes through the pole B of the circle DF, DF will paſs through the pole of BD, (Prop. 4.). And ſince AC is perpendicular to BD, AC will alſo paſs through the pole of BD; wherefore, the pole of the circle BD is in the point where the circles AC, DE interſect, that is, in the point F: The arches FA, FD are therefore

fore quadrants, and likewiſe the arches BD, BE: In the triangle CEF, right angled at the point E, CE is the complement of BC the hypotenuſe of the triangle ABC, EF is the complement of the arch ED, which is the meaſure of the angle ABC, and FC the hypotenuſe of the triangle CEF, is the complement of AC, and the arch AD, which is the meaſure of the angle CFE, is the complement of AB.

But (Prop. 18.) in the triangle CEF, the ſine of the ſide CE is to the radius, as the tangent of the other ſide EF is to the tangent of the angle ECF oppoſite to it; that is, in the triangle ABC, the co-ſine of the hypotenuſe BC is to the radius as the co-tangent of the angle ABC is to the tangent of the angle ACB. Q. E. D.

Cor. 1. Of theſe three, viz. the hypotenuſe and the two angles, any two being given, the third will alſo be given.

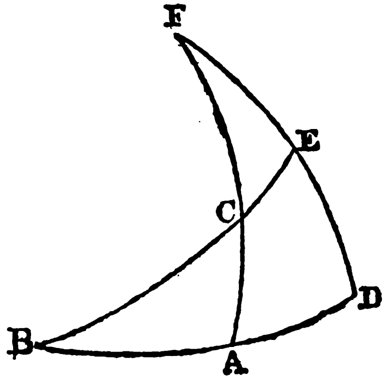
Cor. 2. And ſince by this propoſition the co-ſine of the hypotenuſe BC is to the radius as the co-tangent of the angle ABC to the tangent of the angle ACB, and ſince the radius is to the co-tangent of ACB, as the tangent of ACB to the radius, (Cor. 2. def. 9. Pl. Tr.); therefore *ex æquo*, the co-ſine of the hypotenuſe BC is to the co-tangent of the angle ACB, as the co-tangent of the angle ABC to the radius.

PROP. XXI.

IN right angled ſpherical triangles, the co-ſine of an angle is to the radius, as the tangent of the ſide adjacent to that angle is to the tangent of the hypotenuſe.

The ſame conſtruction remaining: In the triangle CEF, (Prop. 18.) the ſine of the ſide EF is to the radius, as the tangent of the other ſide CE is to the tangent of the angle CFE oppoſite to it, that is, in the triangle ABC, the co-ſine of the angle ABC is to the radius as (the co-tangent of the hypotenuſe BC to the co-tangent of the ſide AB, adjacent to ABC, or as) the tangent of the ſide AB to the tangent of the hypotenuſe, ſince the tangents of two arches are reciprocally proportional to their co-tangents, (Cor. 2. def. 9. Pl. Tr.).

Cor. And ſince by this propoſition the co-ſine of the angle ABC is to the radius, as the tangent of the ſide AB is to the tangent of the hypotenuſe BC; and as the radius is to the co-tangent of BC, ſo is the tangent of BC to the radius; *ex æquo*, the co-ſine of the angle ABC will be to the co-tangent of the hypotenuſe BC, as the tangent of the ſide AB, adjacent to the angle ABC to the radius.



PROP.

PROP. XXII.

IN right angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the hypotenuſe is to the co-ſine of the other ſide.

The ſame conſtruction remaining: In the triangle CEF, the ſine of the hypotenuſe CF is to the radius, as the ſine of the ſide CE to the ſine of the oppoſite angle CFE, (Prop. 19.); that is in the triangle ABC, the co-ſine of the ſide CA is to the radius as the co-ſine of the hypotenuſe BC to the co-ſine of the other ſide BA. Q. E. D.

PROP. XXIII.

IN right angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the angle oppoſite to that ſide is to the ſine of the other angle.

The ſame conſtruction remaining: In the triangle CEF, the ſine of the hypotenuſe CF is to the radius as the ſine of the ſide EF is to the ſine of the angle ECF oppoſite to it, that is, in the triangle ABC, the co-ſine of the ſide CA is to the radius, as the co-ſine of the angle ABC oppoſite to it, is to the ſine of the other angle ACB. Q. E. D.

SOLUTION of the CASES of Righ Angl SPHERICAL TRIANGLES.

PROBLEM.

IN a right angled ſpherical triangle, of the t ſides and three angles, any two being given ſides the right angle, to find the other three.

This problem has ſixteen caſes, the ſolutions of whic contained in the following table, where ABC is any ſph triangle right angled at A.

GIVEN.	SOUGHT.	SOLUTION.
BC and B.	AC	R : ſin. BC : : ſin. B : ſin. AC, (19).
	AB	R : coſ. B : : tan. BC : tan. AB, (21).
	C.	R : coſ. BC : : tan. B : co-t. C, (20).
AC and C.	AB.	R : ſin. AC : : tan. C : tan. AB, (18).
	BC.	coſ. C : R : : tan. AC : tan. BC, (21).
	B,	R : coſ. AC : : ſin. C : coſ. B, (23).
AC and B.	AB.	tan. B : tan. AC : : R : ſin. AB, (18).
	BC.	ſin. B : ſin. AC : : R : ſin. BC, (19).
	C.	coſ. AC : coſ. B : : R : ſin. C, (23).
AC and BC.	AB.	coſ. AC : coſ. BC : : R : coſ. AB, (22).
	B.	ſin. BC : ſin. AC : : R : ſin. B, (19).
	C.	tan. BC : tan. AC : : R : coſ. C. (21).
AB and AC.	BC.	R : coſ. AB : : coſ. AC : coſ. BC, (23).
	B.	ſin. AB : R : : tan. AC : tan. B, (18).
	C.	ſin. AC : R : : tan. AB : tan. C, (18).
B and C.	AB.	ſin. B : coſ C : : R : coſ. AB, (23).
	AC.	ſin. C : coſ. B : : R : coſ. AC, (23).
	BC.	tan. B : co-t. C : : R : coſ. BC, (20).

TAB

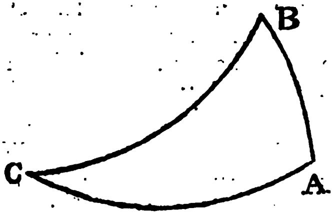


TABLE for determining when the things found in the preceding are leſs than a Quadrant.

The angle or arch found is leſs than 90°.	
When B is leſs than 90°. When BC and B are of the ſame affection. When BC and B are of the ſame affection.	1 2 3
When C is leſs than 90° When AC and C are of the ſame affection. When AC is leſs than 90°.	4 5 6
Ambiguous. Ambiguous. Ambiguous.	7 8 9
When AC and BC are of the ſame affection. When AC is leſs than 90°. When AC and BC are of the ſame affection.	10 11 12
When AB and AC are of the ſame affection. When AC is leſs than 90°. When AB is leſs than 90°.	13 14 14
When C is leſs than 90°. When B is leſs than 90°. When B and C are of the ſame affection.	15 15 16

The caſes marked ambiguous are thoſe in which the thing ſought has two values, and may either be equal to a certain angle, or to the ſupplement of that angle. Of theſe there are three, in all of which the things given are a ſide, and the angle oppoſite to it; and accordingly, it is eaſy to ſhew, that two right angled ſpherical triangles may always be found, that have a ſide and the angle oppoſite to it the ſame in both, but of which the remaining ſides, and the remaining angle of the one, are the ſupplements of the remaining ſides and the remaining angle of the other, each of each.

Though the affection of the arch or angle found may in all the other caſes be determined by the rules in the ſecond of the preceding tables, it may be uſeful to remark, that all theſe rules, except two, may be reduced to one, viz. that *when the thing found by the rules in the firſt table is either a tangent or a co-ſine; and when, of the tangents or co-ſines employed in the computation of it, one only belongs to an obtuſe angle, the angle required is alſo obtuſe.*

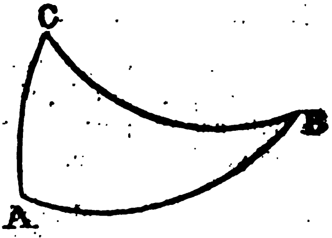
Thus, in the 15th caſe, when coſ. AB is found, if C be an obtuſe angle, becauſe of coſ. C, AB muſt be obtuſe; and in caſe 16. if either B or C be obtuſe, BC is greater than 90°, but if B and C are either both acute, or both obtuſe, BC is leſs than 90°.

It is evident, that this rule does not apply when that which is found is the ſine of an arch; and this, beſides the three ambiguous caſes, happens alſo in other two, viz. the 1ſt and 11th.

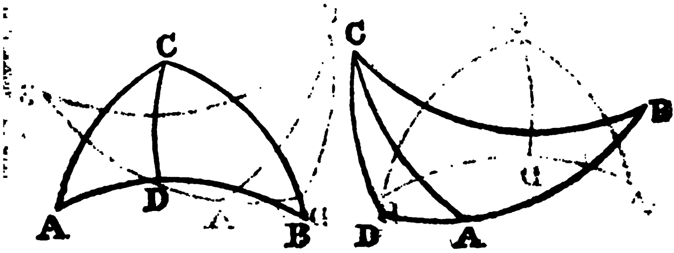
PROP. XXIV.

IN ſpherical triangles, whether right angled or oblique angled, the ſines of the ſides are proportional to the ſines of the angles oppoſite to them.

Firſt, Let ABC be a right angled triangle, having a right angle at A; therefore, (Prop. 19.) the ſine of the hypoteuſe BC is to the radius (or the ſine of the right angle at A) the ſine of the ſide AC the ſine of the angle B. And, in like manner, the ne of BC is to the ſine of he angle A, as the ſine of B to the ſine of the angle ; wherefore (11. 5.) the ſine f the ſide AC is to the ne of the angle B, as the ne of AB to the ſine of the angle C.



Secondly, Let ABC be an oblique angled triangle, the ſine f any of the ſides BC, will be to the ſine of any of the ther two AC, as the ſine of the angle A oppoſite to BC, is the ſine of the angle B oppoſite to AC. Through the oint C, let there be drawn an arch of a great circle CD perendicular upon AB; and in the right angled triangle BCD, Prop. 19.) the ſine of BC is to the radius, as the ſine of D to the ſine of the angle B; and in the triangle ADC, y inverſion, the radius is to the ſine of AC as the ſine the angle A to the ſine of DC: Therefore, *ex æquo pertur* *ate*, the ſine of BC is to the ſine of AC, as the ſine of the ngle A to the ſine of the angle B. Q. E. D.

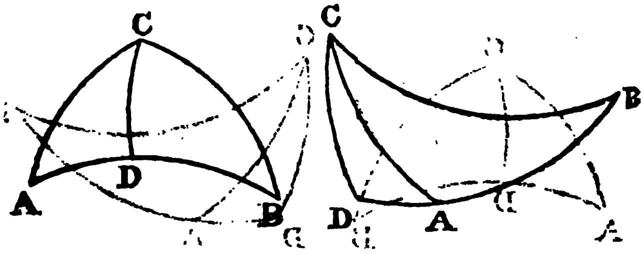


PROP. XXV.

IN oblique angled ſpherical triangles, a perpendicular arch being drawn from any of the angles upon the oppoſite ſide, the co-ſines of the angles at the baſe are proportional to the ſines of the ſegments of the vertical angle.

Let ABC be a triangle, and the arch CD perpendicular to the baſe BA; the co-ſine of the angle B will be to the co-ſine of the angle A, as the ſine of the angle BCD to the ſine of the angle ACD.

For (Prop. 23.) the co-ſine of the angle B is to the ſine of the angle BC, as the co-ſine of the ſide CD is to the radius; and alſo the co-ſine of the angle D to the ſine of the angle ACD in the ſame ratio; therefore by permutation, the co-ſine of the angle B is to the co-ſine of the angle A, as the ſine of the angle BCD to the ſine of the angle ACD. Q. E. D.



PROP. XXVI.

THE ſame things remaining, the co-ſines of the ſides BC, CA, are proportional to the co-ſines of BD, DA, the ſegments of the baſe.

For (Prop. 22.) the co-ſine of BC is to the co-ſine of BD, as the co-ſine of DC to the radius, and the co-ſine of AC to the co-ſine of AD in the ſame ratio: Wherefore, by permutation, the co-ſines of the ſides BC, CA are proportional to the co-ſines of the ſegments of the baſe BD, DA. Q. E. D.

PROP. XXVII.

THE ſame conſtruction remaining; the ſines of BD, DA, the ſegments of the baſe are reciprocally proportional to the tangents of B and A, the angles at the baſe.

For (Prop. 18.) the ſine of BD is to the radius, as the tangent of DC to the tangent of the angle B; and alſo, the radius to the ſine of AD, as the tangent of A to the tangent of DC; Therefore, *ex æquo perturbatè*, the ſine of BD is to the ſine of DA, as the tangent of A to the tangent of B.

PROP. XXVIII.

THE ſame conſtruction remaining, the co-ſines of the ſegments of the vertical angle are reciproprocally proportional to the tangents of the ſides.

For (Prop. 20.) the co-ſine of the angle BCD, is to the radius, as the tangent of CD is to the tangent of BC; and alſo, (Prop. 21. by inverſion), the radius is to the co-ſine of the angle ACD, as the tangent of AC to the tangent of CD: Therefore, *ex æquo perturbatè*, the co-ſine of the angle BCD is to the co-ſine of the angle ACD, as the tangent of AC is to the tangent of BC, Q. E. D.

PROP. XXIX.

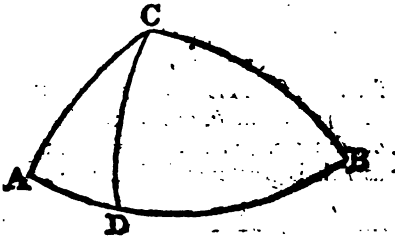
IF from an angle of a ſpherical triangle there be drawn a perpendicular to the oppoſite ſide, or baſe, the rectangle contained by the tangents of half the ſum, and of half the difference of the ſegments of the baſe is equal to the rectangle contained by the tangents of half the ſum, and of half the difference of the two ſides of the triangle.

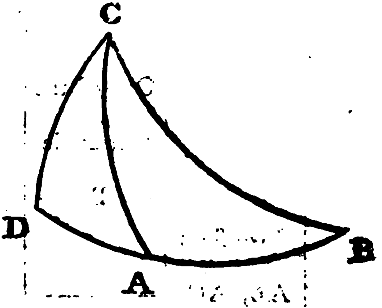
Let ABC be a ſpherical triangle, and let the arch CD be drawn from the angle C at right angles to the baſe AB.

Becauſe (Prop. 26.), coſ. BC : coſ. AC : : coſ. BD : coſ. AD, by compoſition and diviſion, coſ. BC + coſ. AC : coſ. BC — coſ. AC : : coſ. BD + coſ. AD : coſ. BD — coſ. AD. But, (1. Cor. 3. Pl. Trig.), coſ. BC + coſ. AC : coſ. BC — coſ. AC : : cot. $\frac{1}{2}$ (BC+AC) : tan. $\frac{1}{2}$ (BC—AC); and alſo, coſ. BD + coſ. AD : coſ. BD — coſ. AD : : cot. $\frac{1}{2}$ (BD+AD) : tan. $\frac{1}{2}$ (BD—AD), therefore cot. $\frac{1}{2}$ (BC+AC) : tan. $\frac{1}{2}$ (BC—AC) : : cot. $\frac{1}{2}$ (BD+AD) : tan. $\frac{1}{2}$ (BD—AD). And becauſe rectangles of the ſame altitude are as their baſes, therefore, tan. $\frac{1}{2}$ (BC+AC) × cot. $\frac{1}{2}$ (BC+AC) : tan. $\frac{1}{2}$ (BC+AC) × tan. $\frac{1}{2}$ (BC—AC) : : tan. $\frac{1}{2}$ (BD+AD) × cot. $\frac{1}{2}$ (BD+AD) : tan. $\frac{1}{2}$ (BD+AD) × tan. $\frac{1}{2}$ (BD—AD). But tan. $\frac{1}{2}$ (BC+AC) × cot. $\frac{1}{2}$ (BC+AC) = R^2, (2. Cor. def. 9. Pl. Tr.) and alſo, tan. $\frac{1}{2}$ (BD+AD) × cot. $\frac{1}{2}$ (BD+AD) = R^2, therefore (9. 5.) tan. $\frac{1}{2}$ (BD+AD) × tan. $\frac{1}{2}$ (BD—AD) = tan. $\frac{1}{2}$ (BC+AC) × tan. $\frac{1}{2}$ (BC—AC). Q. E. D.

Cor. 1. Becauſe the ſides of equal rectangles are reciprocally proportional, $\tan.\frac{1}{2}(BD+AD) : \tan.\frac{1}{2}(BC+AC) :: \tan.\frac{1}{2}(BC-AC) : \tan.\frac{1}{2}(BD-AD)$.

Cor. 2. Since, when the perpendicular CD falls within the triangle, $BD+AD=AB$, the baſe; and when CD falls without the triangle $BD-AD=AB$, therefore in the firſt caſe, the proportion in the laſt corollary becomes, $\tan.\frac{1}{2}(AB) : \tan.\frac{1}{2}(BC+AC) :: \tan.\frac{1}{2}(BC-AC) : \tan.\frac{1}{2}(BD-AD)$; and in the ſecond caſe, it becomes by inverſion and alternation, $\tan.\frac{1}{2}(AB) : \tan.\frac{1}{2}(BC+AC) :: \tan.\frac{1}{2}(BC-AC) : \tan.\frac{1}{2}(BD+AD$.





SOLUTION of the Cases of Oblique Angled Spherical Triangles.

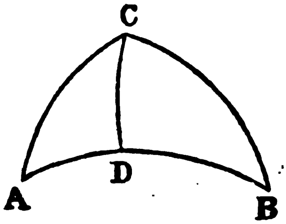
PROBLEM.

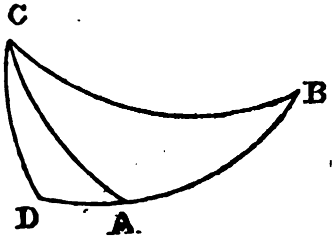
IN any oblique angled ſpherical triangle, of the three ſides and three angles, any three being given, the other three may be found.

In this table, the references (c. 4.), (c. 5.), &c., are to the caſes in the preceding tables.

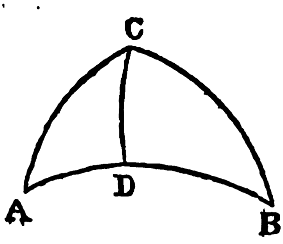
	Given.	Sought.	Solution.
1	Two ſides AB, AC, and the included angle A.	One of the other angles, B.	Let fall the perpendicular CD from the unknown angle not required, on AB. R : coſ. A : : tan. AC : tan. AD, (c. 2.); therefore BD is known, and ſin. BD : ſin. AD : : tan. A : tan. B, (27.); B and A are of the ſame or different affection, according as AB is greater or leſs than AD, (16.)
2		The third ſide BC.	Let fall the perpendicular CD from one of the unknown angles on the ſide AB. R : coſ. A : : tan. AC : tan. AD, (c. 2.); therefore AD is known, and coſ. AD : coſ. BD : : coſ. AC : coſ. BC, (26.); according as the ſegments AD and DB are of the ſame or different affection, AC and CB will be of the ſame or different affection.

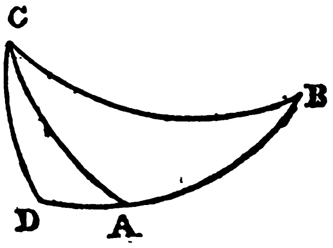
IVEN.	SOUGHT.	SOLUTION.
o angles, nd ACB, and AC, ſide be- en them.	The ſide BC.	From C the extremity of AC next the ſide ſought, let fall the perpendicular CD on AB. R : coſ. AC : : tan. A : cot. ACD, (c. 3.); therefore BCD is known, and coſ. BCD : coſ. ACD : : tan. AC : tan. BC, (28). BC is leſs or greater than 90°, according as the angles A and BCD are of the ſame or different affection.
	The third angle B.	Let fall the perpendicular CD from one of the given angles on the oppoſite ſide AB. R : coſ. AC : : tan. A : cot. ACD, (c. 3.); therefore the angle BCD is given, and ſin. ACD : ſin. BCD : : coſ. A : coſ. B, (25.); B and A are of the ſame or different affection, according as CD falls within or without the triangle, that is, according as ACB is greater or leſs than ACD, (16.)





	GIVEN.	SOUGHT.	SOLUTION.
5	Two ſides AC and BC, and an angle A opposite to one of them, BC.	The angle B opposite to the other given ſide AC.	Sin. BC : ſin. AC :: ſin. A : ſin. B, (24.) The affection of B is ambiguous, unleſs it can be determined by this rule, that according as AC+BC is greater or leſs than 180°, A+B is greater or leſs than 180°, (10.).
6		The angle ACB contained by the given ſides AC and BC.	From ACB the angle ſought draw CD perpendicular to AB; then R : coſ. AC :: tan. A : co-t. ACD, (c. 3.); and tan. BC : tan. AC :: coſ. ACD : coſ. BCD, (28.). ACD±BCD=ACB, and ACB is ambiguous, becauſe of the ambiguous ſign + or —.
7		The third ſide AB.	Let fall the perpendicular CD from the angle C contained by the given ſides upon the ſide AB. R : coſ. A :: tan. AC : tan. AD, (c. 3.); coſ. AC : coſ. BC :: coſ. AD : coſ. BD, (26.) AB=AD±BD, wherefore AB is ambiguous.





GIVEN.

GIVEN.	SOUGHT.	SOLUTION.
	The ſide BC oppoſite to the other given angle A.	Sin. B : ſin. A : : ſin. AC : ſin. BC, (24.); the affection of BC is uncertain, except when it can be determined by this rule, that according as A+B is greater or leſs than 180°; AC+BC is alſo greater or leſs than 180°, (10.).
ro angles A, B, ıd a ſide AC poſite to : of them, B.	The ſide AB adjacent to the given angles A, B.	From the unknown angle C draw CD perpendicular to AB; then R : coſ. A : : tan. AC : tan. AD, (c. 3.); tan. B : tan. A : : ſin. AD : ſin. BD. BD is ambiguous, and therefore AB = AD ± BD may have four values, ſome of which will be excluded by this condition, that AB muſt be leſs than 180°.
	The third angle ACB.	From the angle required, C, draw CD perpendicular to AB. R : coſ. A : : tan. A : co-t. ACD, (c. 3.), coſ. A : coſ. B : : ſin. ACD : ſin. BCD, (25.). The affection of BCD is uncertain, and therefore ACB = ACD ± BCD has four values, ſome of which may be excluded by the condition, that ACB is leſs than 180°.
he three ſides, B, AC, and BC.	One of the angles A.	From C one of the angles not required, draw CD perpendicular to AB. Find an arch E ſuch that tan. $\frac{1}{2}$ AB : tan. $\frac{1}{2}$(AC+BC) : : tan. $\frac{1}{2}$(AC—BC) : tan. $\frac{1}{2}$ E; then, if AB be greater than E, AB is the ſum, and E the difference of AD and DB; but if AB be leſs than E, E is the ſum, and AB the difference of AD, DB, (29.). In either caſe, AD and DB are known, and tan. AC : tan. AD : : R : coſ. A.

	GIVEN.	SOUGHT.	SOLUTION.
12	The three angles A, B, C.	One of the ſides BC.	Suppoſe the ſupplements of the three given angles, A, B, C, to be a, b, c, and to be the ſides of a ſpherical triangle. Find, by the laſt caſe, the angle of this triangle oppoſite to the ſide a, and it will be the ſupplement of the ſide of the given triangle oppoſite to the angle A, that is, of BC, (10.); and therefore BC is found.

In the foregoing table the rules are given for aſcertaining the affection of the arch or angle found, whenever it can be done: Moſt of theſe rules are contained in this one rule, which is of general application, viz. *that when the thing found is either a tangent or a co-ſine, and of the tangents or co-ſines employed in the computation of it, either one or three belong to obtuſe angles, the angle found is alſo obtuſe.* This rule is particularly to be attended to in caſes 5. and 7. where it removes part of the ambiguity.

It may be neceſſary to remark with reſpect to the 11th caſe, that the ſegments of the baſe computed there are thoſe cut off by the neareſt perpendicular; and alſo, that when the ſum of the ſides is leſs than 180°, the leaſt ſegment is adjacent to the leaſt ſide of the triangle; otherwiſe to the greateſt (Prop. 17.).

FINIS.

NOTES

ON THE

ELEMENTS.

NOTES

ON THE

FIRST BOOK OF THE ELEMENTS.

DEFINITIONS.

I.

Book I.

IN the definitions a few changes have been made, of which it is neceſſary that an account ſhould be given. One of theſe changes reſpects the firſt definition, that of a point, which Euclid has ſaid to be, 'That which has no parts, or 'which has no magnitude.' Now, it has been objected to this definition, that it contains only a negative, and that it is not convertible, as every good definition ought certainly to be. That it is not convertible is evident, for though every point is unextended, or without magnitude, yet every thing unextended, or without magnitude, is not a point. To this it is impoſſible to reply, and therefore it becomes neceſſary to change the definition altogether, which is accordingly done here, a point being defined to be, *that which has poſition but not magnitude.* Here the affirmative part includes all that is eſſential to a point, and the negative part excludes every thing that is not eſſential to it. I am indebted for this definition to a friend, by whoſe judicious and learned remarks I have often profited.

II.

After the ſecond definition Euclid has introduced the following, ‘ the extremities of a line are points.’

Now this is certainly not a definition, but an inference from the definitions of a point and of a line. For that which terminates a line can have no breadth, as the line in which it is has none, and it can have no length, as it would not then be a termination, but a part of that which it is ſuppoſed to terminate. The termination of a line can therefore have no magnitude, and having neceſſarily poſition, it is a point. But as it is plain, that in all this we are drawing a conſequence from two definitions already laid down, and not giving a new definition, I have taken the liberty of putting it down as a corrollary to the ſecond definition, and have added, that *the interſections of one line with another are points*, as this affords a good illuſtration of the nature of a point, and is an inference exactly of the ſame kind with the preceding. The ſame thing nearly has been done with the fourth definition, where that which Euclid gave as a ſeparate definition, is made a corollary to the fourth, becauſe it is in fact an inference deduced from comparing the definitions of a ſuperficies and a line.

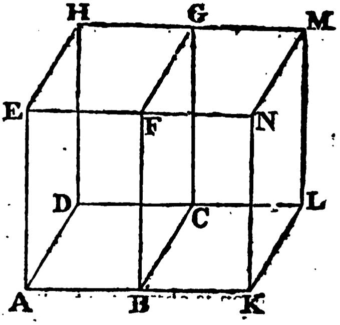
As it is impoſſible to explain the relation of a ſuperficies, a line and a point to one another, and to the ſolid in which they all originate, better than Dr Simſon has done, I ſhall here add, with very little change, the illuſtration given by that excellent Geometer.

“ It is neceſſary to conſider a ſolid, that is, a magnitude which has length, breadth and thickneſs, in order to underſtand aright the definitions of a point, line and ſuperficies ; for theſe all ariſe from a ſolid, and exiſt in it : The boundary, or boundaries which contain a ſolid are called ſuperficies, or the boundary which is common to two ſolids which are contiguous, or which divides one ſolid into two contiguous parts, is called a ſuperficies ; Thus, if BCGF be one of the boundaries which

contain

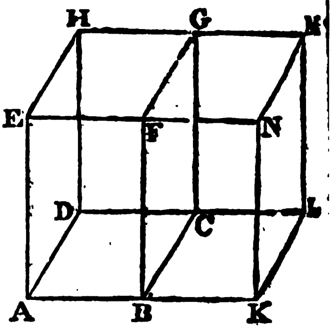
tain the ſolid ABCDEFGH, or which is the common ndary of this ſolid, and the ſolid BKLCFNMG, and is refore in the one as well as the other ſolid, it is called a ſu- ficies, and has no thickneſs: For if it have any, this thick- ſ muſt either be a part of the thickneſs of the ſolid AG, he ſolid BM, or a part of the thickneſs of each of them. annot be a part of the thickneſs of the ſolid BM; becauſe, his ſolid be removed from the ſolid AG, the ſuperficies GF, the boundary of the ſolid AG, remains ſtill the ſame t was. Nor can it be a part of the thickneſs of the ſolid G; becauſe, if this be removed from the ſolid BM, the ſu- ficies BCGF, the boundary of the ſolid BM, does never- leſs remain; therefore the ſuperficies BCGF has no thick- ſ, but only length and breadth.

' The boundary of a ſuperficies is called a line; or a line is common boundary of two ſuperficies that are contiguous, or that which divides one ſuperficies into two contiguous ts: Thus, if BC be one of the boundaries which contain the erficies ABCD, or which is the common boundary of this erficies, and of the ſuperficies KBCL which is contiguous t, this boundary BC is called a line, and has no breadth: , if it have any, this l be part either of the adth of the ſuperficies CD, or of the ſuper- s KBCL, or part of each hem. It is not part of breadth of the ſuper- s KBCL; for, if this rficies be removed from ſuperficies ABCD, the BC which is the boun- of the ſuperficies CD remains the ſame it was. Nor can the adth that BC is ſuppoſed to have, be a part of the breadth he ſuperficies ABCD; becauſe, if this be removed from ſuperficies KBCL, the line BC, which is the boundary of ſuperficies KBCL, does nevertheleſs remain: Therefore line BC has no breadth. And becauſe the line BC is in perficies, and that a ſuperficies has no thickneſs, as was ſhewn;



ſhewn; therefore a line has neither breadth nor thickneſs, but only length.

"The boundary of a line is called a point, or a point is the common boundary or extremity of two lines that are contiguous: Thus, if B be the extremity of the line AB, or the common extremity of the two lines AB, KB, this extremity is called a point, and has no length: For, if it have any, this length muſt either be part of the length of the line AB, or of the line KB. It is not part of the length of KB; for, if the line KB be removed from AB, the point B which is the extremity of the line AB remains the ſame as it was: Nor is it part of the length of the line AB; for, if AB be removed from the line KB, the point B, which is the extremity of the line KB, does nevertheleſs remain: Therefore the point B has no length: And becauſe a point is in a line, and a line has neither breadth nor thickneſs, therefore a point has no length, breadth, nor thickneſs. And in this manner the definitions of a point, line, and ſuperficies are to be underſtood."



III.

Euclid has defined a ſtraight line to be a line which (as we tranſlate it), "lies evenly between its extreme points." This definition is obviouſly faulty, the word evenly ſtanding as much in need of an explanation as the word ſtraight, which it is intended to define. In the original, however, it muſt be confeſſed, that this inaccuracy is at leaſt leſs ſtriking than in our tranſlation; for the word which we render *evenly* is ἐξίσου, *equally*, and is accordingly tranſlated *ex æquo*, and *equaliter* by Gregory and Commandine. The definition, therefore, is, that a ſtraight line is one which lies equally between its extreme points; and if by this we underſtand a line that lies between its extreme points ſo as to be related exactly alike to the ſpace on the one ſide of it, and to the ſpace on the other, we have a definition that is perhaps a little too metaphyſical, but which certainly contains in it the eſſential character of a ſtraight line. That Euclid took the definition in this ſenſe, however, is not certain, becauſe he has not attempted to deduce from it any property whatſoever of a ſtraight line, and indeed, it ſhould ſeem not eaſy to do ſo, without employing ſome reaſonings of a more metaphyſical kind than he has any where admitted into his elements. To ſupply the defects of his definition, he has therefore introduced the Axiom, that *two ſtraight lines cannot incloſe a ſpace*, on which Axiom it is, and not on his definition of a ſtraight line, that his demonſtrations are founded. As this manner of proceeding is certainly not ſo regular and ſcientific as that of laying down a definition, from which the properties of the thing defined may be logically deduced, I have ſubſtituted another definition of a ſtraight line in the room of Euclid's. The definition of a ſtraight line given here is taken from Boſcovich, in his Notes on the Philoſophical Poem of Profeſſor Stay. In one of theſe he ſays, 'Rectam lineam rectæ congruere totam toti in infinitum productam ſi bina puncta unius binis alterius congruant, patet ex ipſa admodum clara rectitudinis idea quam habemus.' (Supplementum in lib. 3. § 550.) Now, that which Mr Boſcovich would conſider as an inference from our idea of ſtraightneſs, ſeems itſelf to be the eſſence of that idea, and to afford the beſt criterion for judging whether any given line be ſtraight or not. From this definition the Axiom above mentioned, viz. That two ſtraight lines cannot incloſe a ſpace, follows as a neceſſary conſequence. For, if two lines incloſe a ſpace, they muſt interſect one another in two points, and yet in the intermediate part muſt not coincide, and therefore by the definition they are not ſtraight lines. It follows in the ſame way, that two ſtraight lines cannot have a common ſegment, or cannot coincide in part, without coinciding altogether.

Since laying down the definition of a ſtraight line, as in the text, I have been favoured by Dr Reid of Glaſgow with the peruſal of a MS. containing many excellent obſervations on the firſt Book of Euclid, ſuch as might be expected from a philoſopher diſtinguiſhed for the accuracy as well as the extent of his knowledge. He there defines a ſtraight line nearly as has been done here, viz. "A ſtraight line is that which cannot meet another ſtraight line in more points than one, otherwiſe they perfectly coincide, and are one and the ſame." Dr Reid alſo contends, that this muſt have been Euclid's own definition; becauſe in the firſt propoſition of the eleventh Book, that author argues, "that two ſtraight lines cannot have a common ſegment, for this reaſon, that a ſtraight line does not meet a ſtraight line in more points than one, otherwiſe they coincide." Whether this amounts to a proof of the definition having been actually Euclid's, I will not take upon me to decide; but it certainly proves that it is a definition which the writings of that geometer ought long ſince to have ſuggeſted to his commentators; and it reminds me, that I might have learnt from theſe writings what I have acknowledged above as derived from a remoter ſource.

There is another characteriſtic, and obvious property of ſtraight lines, by which I have often thought that they might be very conveniently defined, viz that the poſition of the whole of a ſtraight line is determined by the poſition of two of its points, in ſo much that, when two points of a ſtraight line continue fixed, the line itſelf cannot change its poſition. It might therefore be ſaid, that *a ſtraight line is one in which, if the poſition of two points be determined, the poſition of the whole line is determined.* But this definition, though it amounts in fact juſt to the ſame thing with the former, is rather more abſtract than it, and is not ſo eaſily made the foundation of reaſoning; ſo that, after endeavouring as much as poſſible to accommodate it to the underſtanding of beginners, I have found it beſt to lay it aſide, and to adopt the definition given in the text.

V.

The definition of a plane is given from Dr Simſon, Euclid's being liable to the ſame objections with his definition of a ſtraight line, for he ſays, that "a plane ſuperficies is one which lies evenly between its extreme lines." The defects of this definition are completely removed in that which Dr Simſon has given. Another definition different from both might have been adopted, viz. That thoſe ſuperficies are called plane, which are ſuch, that if three points of the one coincide with three points of the other, the whole of the one muſt coincide with the whole of the other. This definition, as it reſembles that of a ſtraight line, already given, migh, perhaps have been introduced with ſome advantage; but as the purpoſes of demonſtration cannot be better anſwered than by that in the text, it has been thought beſt to make no farther alteration.

VI.

In Euclid, the general definition of a plane angle is placed before that of a rectilineal angle, and is meant to comprehend thoſe angles which are formed by the meeting of other lines than ſtraight lines. A plane angle is ſaid to be 'the inclination of two lines to one another which meet together, but are not in the ſame direction.' This definition is omitted here, becauſe that the angles formed by the meeting of curve lines, though they may become the ſubject of geometrical inveſtigation, certainly do not belong to the Elements; for the angles that muſt firſt be conſidered are thoſe made by the interſection of ſtraight lines with one another. The angles formed by the contact or interſection of a ſtraight line and a circle, or of two circles, or two curves of any kind with one another, could produce nothing but perplexity to beginners, and cannot poſſibly be underſtood till the properties of rectilineal angles have been fully explained. On this ground, without conteſting the arguments which Proclus uſes in defence of this definition, I have omitted it. Whatever is not uſeful, ſhould, in explaining the elements of a ſcience, be kept out of ſight altogether; for, if it does not aſſiſt the progreſs of the underſtanding, it will certainly retard it.

AXIOMS.

Among the Axioms there have been made only two alterations. The 10th Axiom in Euclid is, that 'two ſtraight lines 'cannot incloſe a ſpace;' which having become a corollary to our definition of a ſtraight line, ceaſes of courſe to be ranked with ſelf-evident propoſitions. It is therefore removed from among the Axioms, and that which was before the 11th is accounted the 10th.

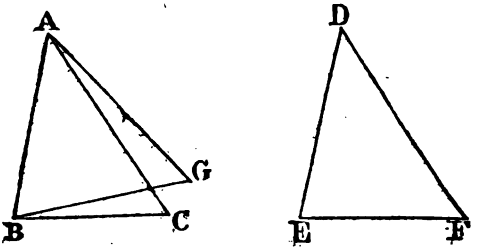
The 12th Axiom of Euclid is, that 'If a ſtraight line 'meets two ſtraight lines, ſo as to make the two interior 'angles on the ſame ſide of it taken together leſs than two 'right angles, theſe ſtraight lines being continually produced, 'ſhall at length meet upon that ſide on which are the angles 'which are leſs than two right angles.' Inſtead of this propoſition, which, though true, is by no means ſelf-evident; another that appeared more obvious, and better entitled to be accounted an Axiom, has been introduced, viz. 'that two ſtraight lines 'cannot be drawn through the ſame point parallel to the ſame 'ſtraight line, without coinciding with one another.' On this ſubject, however, a fuller explanation is neceſſary, for which ſee the note on the 29th Prop.

PROP. IV. and VIII. B. I.

The fourth and eighth propoſitions of the firſt book are the foundation of all that follows with reſpect to the compariſon of triangles. They are demonſtrated by what is called the method of ſupraposition, that is, by laying the one triangle upon the other, and proving that they muſt coincide. To this ſome objections have been made, as if it were ungeometrical to ſuppoſe one figure to be removed from its place and applied to another figure. 'The laying,' ſays Mr Thomas Simpſon in his Elements, 'of one figure upon another, whatever evidence it may afford, is a *mechanical* conſideration, and depends on no poſtulate.' It is not clear what Mr Simpſon meant here by the word *mechanical*; but he probably intended only to ſay, that the method of ſupraposition involves the idea of motion, which belongs rather to mechanics than geometry; for, I think it is impoſſible that ſuch a Geometer as he was could mean to aſſert, that the evidence derived from this method is like that which ariſes from the uſe of inſtruments, and of the ſame kind with what is furniſhed by experience and obſervation. The demonſtrations of the fourth and eighth, as they are given by Euclid, are as certainly a proceſs of pure reaſoning, depending ſolely on the idea of equality, as eſtabliſhed in the 8th Axiom, as any thing in geometry. But, if ſtill the removal of the triangle from its place be conſidered as creating a difficulty, and as inelegant, becauſe it involves an idea, that of motion, not eſſential to geometry, this defect may be entirely remedied, provided that, to Euclid's three poſtulates, we be allowed to add the following, viz. *That if there be two equal ſtraight lines, and if any figure whatſoever be conſtituted on the one, a figure every way equal to it may be conſtituted on the other.* Thus, if AB and DE be two equal ſtraight lines, and ABC a triangle on the baſe AB, a triangle DEF every way equal to ABC may be ſuppoſed to be conſtituted on DE as a baſe. By this it is not meant to aſſert that the method of deſcribing the triangle DEF is actually known, but merely that the triangle DEF may be conceived to exiſt in all reſpects equal to the triangle ABC. Now, there is no truth whatſoever that is better entitled than this to be ranked among the Poſtulates or Axioms of geometry; for the ſtraight lines AB and DE being every way equal, there can be nothing belonging to the one that may not alſo belong to the other.

On the ſtrength of this poſtulate the fourth Prop. is thus demonſtrated.

If ABC, DEF be two triangles, ſuch that the two ſides AB and AC of the one are equal to the two ED, DF of the other, and the angle BAC, contained by the ſides AB, AC of the one, equal to the angle EDF, contained by the ſides ED, DF of the other; the triangles ABC and EDF are every way equal.



On AB let a triangle be conſtituted every way equal to the triangle DEF; then, if this triangle coincide with the triangle ABC, it is evident that the propoſition is true, for it is equal to DEF by hypotheſis, and to ABC, becauſe it coincides with it; wherefore ABC, DEF are equal to one another. But if it does not coincide with ABC, let it have the poſition ABG; and firſt, let G not fall on AC; then the angle BAG is not equal to the angle BAC. But the angle BAG is equal to the angle EDF, therefore EDF and ABC are not equal, and they are alſo equal by hypotheſis, which is impoſſible. Therefore the point G muſt fall upon AC; now, if it fall upon AC but not at C, then AG is not equal to AC; but AG is equal to DF, therefore DF and AC are not equal, and they are alſo equal by ſuppoſition, which is impoſſible. Therefore G muſt coincide with C, and the triangle AGB with the triangle ACB. But AGB is every way equal to DEF, therefore ACB and DEF are alſo every way equal. Q.E.D.

In the ſame manner may the 8th be demonſtrated, and even with more facility, ſo that theſe two fundamental Theorems may be proved without having recourſe to the method of ſuprapoſition. Such demonſtrations, it muſt, however, be acknowledged, treſpaſs againſt a rule which Euclid has uniformly adhered to throughout the Elements, except where he was forced by neceſſity to depart from it. This rule is, that nothing is ever ſuppoſed to be done, the manner of doing which has not been already taught, and the conſtruction derived either directly from the three poſtulates laid down in the beginning, or from ſome problems already reduced to thoſe poſtulates. Now, this rule is not eſſential to geometrical demonſtration, where, for the purpoſe of diſcovering the properties of figures, we are certainly at liberty to ſuppoſe any figure to be conſtructed, or any line to be drawn, the exiſtence of which does not involve an impoſſibility. The only uſe, therefore, of Euclid's rule is to guard againſt the introduction of impoſſible hypotheſes, or the taking for granted that a thing may exiſt which in fact implies a contradiction; from ſuch ſuppoſitions, falſe concluſions might, no doubt, be deduced, and this rule is therefore uſeful as far as it anſwers the purpoſe of excluding them. But the foregoing poſtulatum could never lead to ſuppoſe the actual exiſtence of any thing that is impoſſible; for it only ſuppoſes the exiſtence of a figure equal and ſimilar to one already exiſting, but in a different part of ſpace from it, or, to ſpeak more preciſely, having one of its ſides in an aſſigned poſition. Unleſs therefore there be an impoſſibility in the exiſtence of the one of theſe figures, there can be none in that of the other. This new poſtulate might, therefore, it ſhould ſeem, be introduced with good effect into the Elements of Geometry; but to have adopted it here would have led us too far from the text of Euclid, and it is ſufficient for the preſent purpoſe to have pointed it out.

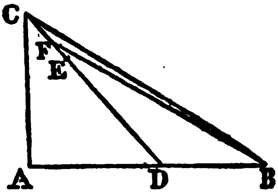
PROP. VII.

Dr Simſon has very properly changed the enunciation of this propoſition, which, as it ſtands in the original, is conſiderably embarraſſed and obſcure. His enunciation, with very little variation, is retained here.

PROP. XXI.

It is eſſential to the truth of this propoſition, that the ſtraight lines drawn to the point within the triangle be drawn from the two extremities of the baſe; for, if they be drawn from other points of the baſe, their ſum may exceed the ſum of the ſides of the triangle in any ratio that is leſs than that of two to one. This is demonſtrated by Pappus Alexandrinus in the 3d Book of his *Mathematical Collections*, but the demonſtration is of a kind that does not belong to this place. If it be required ſimply to ſhew, that in certain caſes the ſum of the two lines drawn to the point within the triangle may exceed the ſum of the ſides of the triangle, the demonſtration is eaſy, and is given nearly as follows by Pappus, and alſo by Proclus, in the 4th Book of his Commentary on Euclid.

Let ABC be a triangle, having the angle at A a right angle; let D be any point in AB; join CD, then CD will be greater than AC, becauſe in the triangle ACD, the angle CAD is greater than the angle ADC. From DC cut off DE equal to AC; biſect AE in F, and join BF; BF and FD are greater than BC and CA.



Becauſe CF is equal to FE, CF and FB are equal to EF and FB, but CF and FB are greater than BC, therefore EF and FB are greater than BC. To EF and FB add ED, and to BC add AC, which is equal to ED by conſtruction, and BF and FD will be greater than BC and CA. Q. E. D.

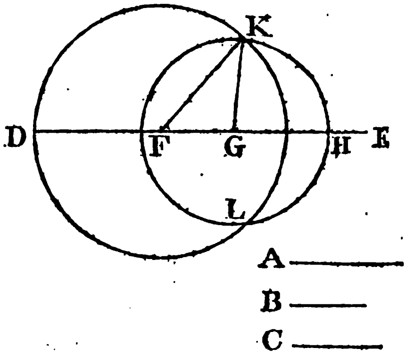
It is evident, that if the angle BAC be obtuſe, the ſame reaſoning may be applied.

This propoſition is a ſufficient vindication of Euclid for having demonſtrated the 21ſt propoſition, which ſome affect to conſider as ſelf-evident; for it proves, that the circumſtance on which the truth of that propoſition depends is not obvious, nor that which at firſt ſight it is ſuppoſed to be, viz. that of the one triangle being included within the other. For this reaſon I cannot agree with M. Clairault, that Euclid demonſtrated this propoſition only to avoid the cavils of the Sophiſts. But I muſt, at the ſame time, obſerve, that what the former has ſaid on the ſubject has certainly been miſunderſtood, and in one reſpect, unjuſtly cenſured by Dr Simſon. The exact tranſlation of his words is as follows: "If Euclid has taken the trouble "to demonſtrate, that a triangle included within another has "the ſum of its ſides leſs than the ſum of the ſides of the triangle in which it is included, we are not to be ſurpriſed. "That geometer had to do with thoſe obſtinate Sophiſts, who "made a point of refuſing their aſſent to the moſt evident "truths," &c. (Elemens de Geometrie par M. Clairault. Pref.)

Dr Simſon ſuppoſes M. Clairault to mean by the propoſition which he enunciates here, that when one triangle is included in another, the ſum of the two ſides of the included triangle is neceſſarily leſs than the ſum of the two ſides of the triangle in which it is included, whether they be on the ſame baſe or not. Now, this is not only not Euclid's propoſition, as Dr Simſon remarks, but it is not true, and is directly contrary to what has juſt been demonſtrated from Proclus. But, the fact ſeems to be, that M. Clairault's meaning is entirely different, and that he intends to ſpeak not of two of the ſides of a triangle, but of all the three, ſo that his propoſition is, "that when one triangle is included within "another, the ſum of all the three ſides of the included tri- "angle is leſs than the ſum of all the three ſides of the other," and this is without doubt true, though, I think, by no means ſelf-evident. It muſt be acknowledged alſo, that it is not exactly Euclid's propoſition, which, however, it comprehends under it, and is the general theorem, of which the other is only a particular caſe. Therefore, though M. Clairault may be blamed for maintaining that to be an Axiom which requires demonſtration, yet he ought not to be accuſed of miſtaking in this inſtance a falſe propoſition for a true one.

PROP. XXII.

Thomas Simpſon in his Elements has objected to Euclid's demonſtration of this propoſition, becauſe it contains no proof, that the two circles made uſe of in the conſtruction of the Problem muſt cut one another; and Dr Simſon, on the other hand, always unwilling to acknowledge the ſmalleſt blemiſh in the works of Euclid, contends, that the demonſtration is perfect. The truth, however, certainly is, that the demonſtration admits of ſome improvement; for the limitation that is made in the enunciation of any Problem ought always to be ſhewn to be neceſſarily connected with the conſtruction of it, and this is what Euclid has neglected to do in the preſent inſtance. The defect may eaſily be ſupplied, and Dr Simſon himſelf has done it in effect in his note on this propoſition, though he denies it to be neceſſary.



Becauſe that of the three ſtraight lines DF, FG, GH, any two are greater than the third, by hypotheſis, FD is leſs than FG and GH, that is, than FH, and therefore the circle deſcribed from the centre F with the diſtance FD muſt meet the line FE between F and H; and for the like reaſon, the circle deſcribed from the centre G at the diſtance GH, muſt meet DG between D and G, and therefore, the one of theſe circles cannot be wholly within the other. Neither can the one be wholly without the other, becauſe DF and GH are greater than FG; the two circles muſt therefore interſect one another.

PROP. XXIV.

The ſubject of parallel lines is one of the moſt difficult in the Elements of Geometry. It has accordingly been treated of in a great variety of different ways, of which, perhaps, there is none that can be ſaid to have given entire ſatisfaction. The difficulty conſiſts in *converting* the 27th and 28th of Euclid, or in demonſtrating, that parallel ſtraight lines, or ſuch as do not meet one another, when they meet a third line, make the alternate angles with it equal, or which comes to the ſame, are equally inclined to it, and make the exterior angle equal to the interior and oppoſite. In order to demonſtrate this propoſition, Euclid aſſumed it as an Axiom, that 'if a ſtraight line meet two ſtraight lines, ſo as to make the 'interior angles on the ſame ſide of it leſs than two right 'angles, theſe ſtraight lines being continually produced, will 'at length meet on the ſide on which the angles are that are 'leſs than two right angles.' This propoſition, however, is not ſelf-evident, and ought the leſs to be received without proof, that, as Proclus has obſerved, the converſe of it is a propoſition that confeſſedly requires to be demonſtrated. For the converſe of it is, that two ſtraight lines which meet one another make the interior angles, with any third line, leſs than two right angles; or, in other words, that the two interior angles of any triangle are leſs than two right angles, which is the 17th of the Firſt Book of the Elements; and it ſhould ſeem, that a propoſition can never rightly be taken for an Axiom, of which the converſe requires a demonſtration.

This defect in Euclid is, therefore, abundantly evident, but the manner of correcting it is by no means obvious, as will appear from a review of the methods chiefly employed for that purpoſe.

Theſe have been of three ſorts: 1. A new definition of parallel lines: 2. A new manner of reaſoning on the properties of ſtraight lines without any new Axiom: 3. The introduction of a new Axiom, leſs exceptionable than Euclid's.

1. One of the definitions that has been ſubſtituted for Euclid's is, that ſtraight lines are parallel, which preſerve always the ſame diſtance from one another, by the word diſtance being underſtood, a perpendicular drawn to one of the lines from any point whatever in the other. If theſe perpendiculars be every where of the ſame length, the ſtraight lines are called parallel. This is the definition given by Wolfius, by Boſcovich, and by Thomas Simpſon, in the firſt edition of his Elements. It is, however, a very faulty definition, for it conceals an Axiom in it, and takes for granted a property of ſtraight lines, that ought either to be laid down as ſelf evident, or demonſtrated, if poſſible, as a Theorem. Thus, if

from

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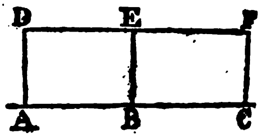
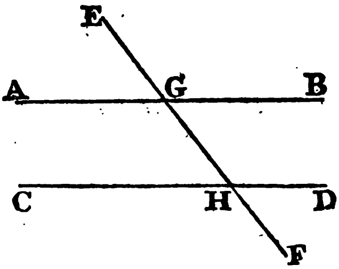
from the three points A, B, and C of the ſtraight line AC, perpendiculars AD, BE, CF be drawn all equal to one another, it is implied in the definition, that the points D, E and F are in the ſame ſtraight line, which, though it be true, it was not the buſineſs of the definition to inform us of. Two perpendiculars, as AD and CF, are alone ſufficient to determine the poſition of the ſtraight line DF, and therefore the definition ought to be, “ that two ſtraight lines are parallel, “ when there are two points in the one, from which the perpendiculars drawn to the other are equal, and on the ſame “ ſide of it.”



This is the definition of parallels which M. D'Alembert ſeems to prefer to all others; but he acknowledges, and very juſtly, that it ſtill remains a matter of difficulty to demonſtrate, that all the perpendiculars drawn from the one of theſe lines to the other are equal. (*Encyclopedie, Art. Parallele.*)

Another definition that has been given of parallels is, that they are lines which make equal angles with a third line, toward the ſame parts, or ſuch as make the exterior angle equal to the interior and oppoſite. Varignon, Bezout, and ſeveral other mathematicians have adopted this definition, which, it muſt be acknowledged, is a perfectly good one, if it be underſtood by it, that the two lines, called parallel, are ſuch as make equal angles with a certain third line, but not with any line that falls upon them. For, if this laſt is ſuppoſed, we, in fact, involve a Theorem in the definition, viz. That if AB and CD make equal angles with GH, they will do ſo alſo with any other line whatſoever, which is a propoſition to be proved. But, the definition, when thus explained, has no advantage whatever above Euclid's; and it ſtill remains to demonſtrate, 1. That ſtraight lines which are equally inclined to a certain line that falls upon them, muſt be equally inclined to all the others that fall upon them; and alſo



ſo, 2. That ſtraight lines which do not meet when produced, muſt make equal angles with every line that falls upon them. On the whole, therefore, we may conclude, that of the three definitions that may be given of parallels, Euclid's, and the other two juſt mentioned, no one, as to the object of facilitating the demonſtration of the properties of ſuch lines, has any advantage above the reſt.

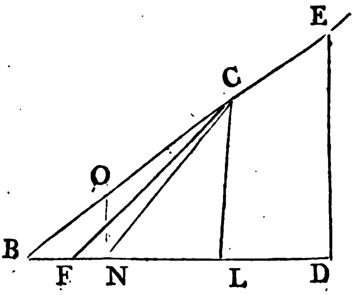
Next, with reſpect to thoſe who have ſought to demonſtrate the properties of parallel lines by help of any of the preceding definitions, without any new Axiom, all their attempts, as far as I know, have been unſucceſsful. Of theſe I ſhall, however, mention only two, the one the moſt ancient, and the other the moſt recent of which I have any information.

The firſt is by Ptolemy the aſtronomer, who, as Proclus informs us, wrote an entire Book in proof of the propoſition which Euclid has conſidered as an Axiom, concerning the meeting of ſuch lines, as make with another line the interior angles leſs than two right angles. Proclus has preſerved an account of this work, in the fourth book of his Commentaries, from which it appears, that Ptolemy had endeavoured to give his demonſtrations without introducing any new Axiom; and it is remarkable, that he reaſons there exactly on the principle which the moderns have diſtinguiſhed by the name of the *ſufficient reaſon.* To prove, that if two parallel ſtraight lines, AB and CD, be cut by a third line EF, in G and H, the two interior angles AGH, CHG will be equal to two right angles, he reaſons thus: If the angles AGH, CHG be not equal to two right angles, let them, if poſſible, be greater than two right angles; then, becauſe the lines AG and CH are not more parallel than the lines BG and DH, the angles BGH, DHG are alſo greater than two right angles. Therefore, the four angles AGH, CHG, BGH, DHG are greater than four right angles; and they are alſo equal to four right angles, which is abſurd. In the ſame manner it is ſhewn, that the angles AGH, CHG cannot be leſs than two right angles. Therefore they are equal to two right angles.

Proclus objects to this reaſoning, and it certainly cannot be admitted as concluſive. For why are we to ſuppoſe that the interior angles which the parallels make with the line cutting them, are either in every caſe greater than two right angles, or in every caſe leſs than two right angles? For any thing that we are yet ſuppoſed to know, they may be ſometimes greater than two right angles, and ſometimes leſs, and therefore we are not entitled to conclude, becauſe the angles AGH, CHG are greater than two right angles, that therefore the angles BGH, DHG are alſo neceſſarily greater than two right angles. It may ſafely be aſſerted, therefore, that Ptolemy has not ſucceeded in his attempt to demonſtrate the properties of parallel lines without the aſſiſtance of a new Axiom.

The other attempt to demonſtrate the ſame propoſition without the aſſiſtance of a new Axiom is, by a modern geometer, Franceſchinis, Profeſſor of Mathematics in the Univerſity of Bologna, in an eſſay, which he entitles, *La Teoria delle parallele rigoroſamente dimoſtrata*, printed in his *Opuſcoli Mathematici*, at Baſſano in 1787.

The difficulty is there reduced to a propoſition nearly the ſame with this, That if BE make an acute angle with BD, and if DE be perpendicular to BD at any point, BE and DE, if produced, will meet. To demonſtrate this, it is ſuppoſed, that BO, BC are two parts taken in BE, of which BC is greater than BO, and that the perpendiculars ON, CL are drawn to BD; then ſhall BL be greater than BN. For, if not, that is, if the perpendicular CL falls either at N, or between B and N, as at F; in the firſt of theſe caſes the angle CNB is equal to the angle ONB, becauſe they are both right angles, which is impoſſible; and, in the ſecond, the two angles CFN, CNF, of the triangle CNF, exceed two right angles. Therefore, adds our author, ſince, as BC increaſes, BL alſo increaſes, and ſince BC may be increaſed without limit, ſo BL may become greater than any given line, and therefore may be greater than BD; wherefore, ſince the perpendiculars to BD from points beyond D meet BC, the perpendicular from D neceſſarily meets it. Q. E. D.



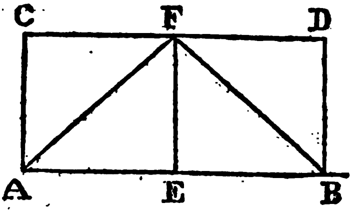
Now, though at firſt ſight this reaſoning may appear ſound, it will be found, on examination, to be entirely inconcluſive. For, unleſs it be proved, that whatever multiple BC is of BO, the ſame is BL of BN, the indefinite increaſe of BC does not neceſſarily imply the indefinite increaſe of BL, or that BL may be made to exceed BD. On the contrary, BL may always increaſe, and yet may do ſo in ſuch a manner as never to exceed BD: In order that the demonſtration ſhould be concluſive, it would be neceſſary to ſhew, that when BC increaſes by a part equal to BO, BL increaſes always by a part equal to BN; but to do this will be found to require the knowledge of thoſe very properties of parallel lines that we are ſeeking to demonſtrate.

The foregoing propoſition, therefore, is not demonſtrated; if it were, the doctrine of parallel lines would be ſubject to no farther difficulty, as will appear from the demonſtration of Clavius.

Clavius is one of the geometers who has treated moſt fully of this ſubject, but it may be doubted, whether he conſidered his demonſtration as being founded on a new Axiom, or not. The propoſition that ſerves as the foundation of his reaſoning he certainly does not lay down entirely as ſelf evident, for he offers a ſort of metaphyſical proof of it, by which he endeavours to connect it with Euclid's definition of a ſtraight line. In this he is not ſucceſsful, and, by what he ſays at the concluſion of the whole, he appears to have been willing to wave this proof, and to conſider the ſaid propoſition as an Axiom; for he ſays, "*Quamvis autem, conceſſo principio* "*noſtro optime a nobis demonſtratum ſit tertium decimum hoc* "*Axioma, (ſc. Euclidis,)*" *&c.*; where he certainly ſpeaks of the principle of his demonſtration as a thing taken for granted. It is this, "That a line of which the points are all "equidiſtant from a certain ſtraight line in the ſame plane "with it, is itſelf a ſtraight line." From this propoſition, which, however, ought not in ſtrictneſs to be held an Axiom, it follows, that if two equal perpendiculars be drawn to any ſtraight line on the ſame ſide of it, the ſtraight line joining their extremities will be equidiſtant from the former line. For, if not, it is plain from it, that two ſtraight lines may comprehend a ſpace.

Book I. Hence, if through the extremities C and D of two perpendiculars, AC and BD, to the ſame ſtraight line AB, a ſtraight line CD be drawn, CD ſhall make right angles with the perpendiculars AC and BD.

For, if AB be biſected in E, and EF be drawn perpendicular to AB meeting CD in F; and if AF, BF be drawn, becauſe the ſides AE, EF of the triangle AEF are equal to the ſides BE, EF of the triangle BEF, and becauſe they contain the equal angles AEF, BEF, each of which is a right angle, therefore the baſe AF is equal to the baſe BF, and the angle EFA to the angle EFB, as alſo the angle EAF to the angle EBF. Now, ſince the whole angle CAE is equal to the whole DBE, and the part EAF to the part EBF, therefore the remainder FAC is equal to the remainder FBD. Therefore, ſince AC is equal to BD, and AF to BF; and, ſince the angles CAF, DBF are alſo equal, the triangles ACF, BDF are equal, the angle ACF to the angle BDF, and the angle AFC to the angle BFD. Now, it has been ſhewn, that the angle AFE is equal to the angle BFE, therefore the whole angle CFE is equal to the whole angle DFE, and therefore CFE, DFE are each of them right angles. But, becauſe AC and EF are equal perpendiculars drawn to the line AE, and CF paſſes through their extremities, the angles ACF, CFE will be ſhewn to be equal, in the ſame way that the angles ACF and FDB have juſt been ſhewn to be equal. But CFE is a right angle, therefore ACF is alſo a right angle. For the ſame reaſon BDC is a right angle.



Hence, laſtly, the demonſtration of Euclid's 12th Axiom, viz. that when a ſtraight line, falling on two other ſtraight lines, makes the internal angles, on the ſame ſide of it, leſs than two right angles, the two lines will meet, if continually produced towards that ſide on which the two angles are that are leſs than two right angles.

et AC and BD be two ſtraight lines, on which AB
and makes
ngles CAB,
) leſs than
ight angles,
and BD will
, if produ-
on the ſide
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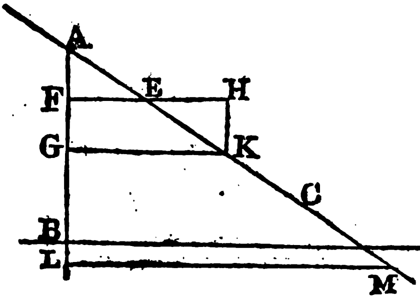
ſt, let ABD
right angle,
therefore
an acute
. In AC
ny point E,

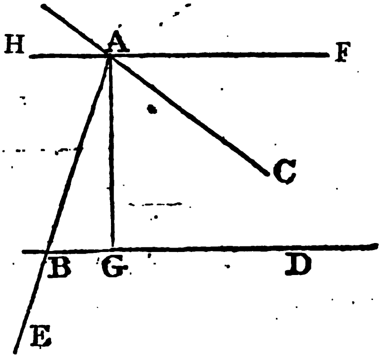


raw EF perpendicular to AB ; alſo, let AL be a multiple
F ſuch as to exceed AB, and let AM be the ſame mul-
of AE that AL is of AF. Take FG equal to AF,
K equal to EA, produce FE to H, ſo that EH may be
to EF, and join HK ; join alſo LM.

cauſe AE is equal to EK, FE to EH, and the angle
to the angle KEH, the triangles AFE, KHE are equal,
aſe HK to the baſe AF, and the angle EHK to the angle
.. But AFE is a right angle, wherefore EHK is alſo a
angle, and HK being equal to AF, that is, to FG, KGE
ight angle, by the laſt propoſition, and KG a perpen-
ar to AB ; and, as the ſame may be proved of all the
joining the correſponding points of diviſion in AB and
it is true alſo of LM, and therefore LM is perpendicu-
LA. But BD is alſo perpendicular to LA, therefore
nd BD are parallel, and the whole of the line LM is
fore on the ſame ſide of BD, namely, on the ſide of it
ſite to A, therefore the line MA, that is, AC, muſt cut
ne BD.

xt, Let the angles BAC and ABD, which the two
make with the third line, be neither of them right angles.
ugh A draw HAF making the angle BAF equal to the
EBD, and make AG perpendicular to BD; and, be-
the angles BAC, ABD are leſs than two right angles,
he angles EBD, ABD equal to two right angles, the

angles EBD and ABD are together greater than the angles CAB, ABD, and taking away the angle ABD, there remains the angle EBD greater than the angle BAC; but EBD is equal to BAF, therefore BAF is greater than BAC, and AC falls between AF and BD. Now, becauſe the angle EBD is equal to the angle BAF, BD is parallel to AF; and becauſe AG is at right angles to BD, it is alſo at right angles to AF; for, if not, the angle GAF will either be acute or obtuſe, but it is not acute, for then, by the laſt caſe, the ſtraight lines AC and BD would meet toward C and D, which is abſurd, for AF and BD are parallel. Neither is the angle GAF obtuſe, for the angle HAG would then be acute, and the ſtraight lines AH and GB would meet, if produced toward B and H. Since the angle GAF, therefore, is a right angle, the angle GAC is leſs than a right angle, and therefore, by the laſt caſe, AC and GD will meet, if produced continually toward C and D. Q. E. D.

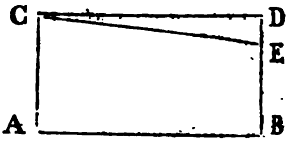


It is thus that Clavius demonſtrates Euclid's Axiom, and, of conſequence, the whole doctrine of parallel lines, in a manner altogether unexceptionable, if the principle be taken for granted, "that a line, of which the points are all equidiſtant "from a certain ſtraight line, in the ſame plane with it, is itſelf a ſtraight line." A demonſtration, not very different from the preceding, is given by Dr Simſon in the ſecond, and the ſubſequent editions of his Euclid. In his firſt edition he followed a method which he admits not to be ſtrictly demonſtrative, grounded on this, than when two ſtraight lines cut one another, they may be produced till their diſtance become greater than any given line, a principle which Proclus has alſo employed for the ſame purpoſe. In his other demonſtration he differs from Clavius, firſt, as to the principle which he takes for granted; which is, "that a ſtraight "line cannot firſt come nearer to another ſtraight line, and "then go farther from it without meeting it;" where, by *coming nearer*, is underſtood, according to a previous definition, the diminution of the perpendiculars drawn from the one line upon the other; and the contrary by *going farther off.* This Axiom is, no doubt, more readily aſſented to than that of Clavius, and muſt be regarded as the more fundamental propoſition of the two, though there be, in fact, no great difference between them. It is evident how it may be employed to demonſtrate Clavius's firſt Theorem, viz. That if the perpendiculars drawn upon one ſtraight line from two points in another be equal, all the perpendiculars drawn to the firſt of theſe lines from the points of the ſecond are equal to one another. For otherwiſe the perpendiculars muſt firſt increaſe and then decreaſe, which, by the Axiom, is impoſſible. After this the demonſtration proceeds in the ſame manner with that which has juſt been given, except that in ſome parts the proof is rendered direct where Clavius has it indirect, as in the ſecond caſe of the laſt propoſition. On the whole, however, the two demonſtrations are ſo nearly alike, that as the one of them has been fully explained, the other does not require a particular conſideration.

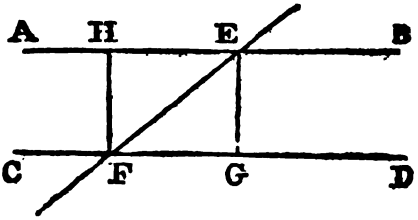
Theſe, then, are ſome of the moſt remarkable demonſtrations of the properties of parallel lines, that have been given, ſo far as I know, by ancient or modern geometers. I ſhall mention only one more, which I cannot help regarding as ſuperior in ſimplicity and neatneſs to any of them. It is that which is given by Mr Thomas Simpſon, in the ſecond edition of his Elements, which is very different from that of the firſt edition, where he lays down the faulty definition of parallel lines that was already taken notice of. The Axiom employed by Simpſon is eſſentially the ſame with that which was laſt mentioned, but it is preſented in a ſimpler form. It is, that "if two points in a ſtraight line are poſited at unequal "diſtances from another ſtraight line in the ſame plane, thoſe "two lines being indefinitely produced on the ſide of the "leaſt diſtance will meet one another."

By help of this Axiom it is eaſy to prove, that if two ſtraight lines AB, CD be parallel, the perpendiculars to the one, terminated by the other, are all equal, and are alſo perpendicular to both the parallels. That they are equal is evident, otherwiſe the lines would meet by the Axiom: That they are perpendicular to both is demonſtrated thus:

If AC and BD, which are perpendicular to AB, and equal to one another, be not alſo perpendicular to CD, from C let CE be drawn at right angles to BD. Then, becauſe AB and CE are both perpendicular to BD, they are parallel, and therefore, the perpendiculars AC and BE are equal. But AC is equal to BD, (by hypotheſis), therefore BE and BD are equal, which is impoſſible; BD is therefore at right angles to CD.



Hence the propoſition, that "if a ſtraight line fall on two "parallel lines, it makes the alternate angles equal," is eaſily derived. Let FH and GE be perpendicular to CD, then they will be parallel to one another, and alſo at right angles to AB, and therefore FG and HE are equal to one another, by the laſt propoſition. Wherefore in the triangles EFG, EFH, the ſides HE and EF are equal to the ſides GF and FE, each to each, and alſo the third ſide HF to the third ſide EG, therefore the angle HEF is equal to the angle EFG, and they are alternate angles. Q. E. D.



This method of treating the doctrine of parallel lines is extremely plain and conciſe, and perhaps does not admit of any improvement, except that of putting the Axiom on which it is founded into a more ſimple form: It might, for example, be expreſſed thus: "If two ſtraight lines interſect one another, and if from any two points of the one, perpendiculars be drawn to the other; the perpendicular that is near-

'er to the point of interſection is leſs than that which is more 'remote."

The demonſtration founded on this Axiom, and conducted in the manner juſt explained, I am diſpoſed to think preferable to any that has yet been given. If I have not followed this method, it is becauſe I wiſhed to preſerve the text of Euclid with the leaſt alteration poſſible. I therefore aſſumed as an Axiom, a propoſition that is not perhaps ſo obvious a property of ſtraight lines as that which has juſt been ſtated, but which is certainly much more ſo than Euclid's, and one which I know from experience that beginners find no difficulty in comprehending, or in admitting to be true. By means of it the 29th, and alſo what was formerly the 12th Axiom, are demonſtrated without changing any thing in the ſeries of Euclid's propoſitions.

From the detail that has juſt been given, it is evident, that to demonſtrate the properties of parallel lines without having recourſe to ſome Axiom, or which is the ſame thing, without aſſuming ſome property of ſtraight lines, not contained in the definition of a ſtraight line, is ſtill a deſideratum in elementary geometry. And, if we conſider how much ſkill and ingenuity have in the courſe of many ages been applied to this inveſtigation; and alſo reflect, that the thing ſought for belongs to the very rudiments of the ſcience, and therefore, if it exiſts at all, can be at no great diſtance, we ſhall be inclined to conſider the diſcovery of it, as a problem in geometry never likely to be reſolved. At the ſame time, it appears extraordinary, that the definition of a ſtraight line, if it is complete, ſhould not lead us to the knowledge of every property of ſuch a line, without the aſſumption of any thing not contained in it. Why ought not the propoſition, for inſtance, that has been juſt ſtated as an Axiom, that "if two ſtraight lines "interſect one another, and if from any two points in the one "perpendiculars be drawn to the other, the perpendicular "nearer to the point of interſection will be leſs than that "which is more remote," to be capable of demonſtration, or of being deduced from the definition of a ſtraight line? If there be nothing obſcure or imperfect in our notions of a ſtraight line, of a perpendicular, or of the interſection of two ſtraight lines, from whence can it poſſibly ariſe that we are unable to demonſtrate this propoſition? It was no way wonderful, that, when Euclid gave his vague and obſcure definition of a ſtraight line, he ſhould not be able to demonſtrate even the moſt ſimple property of rectilineal figures, without the aſſumption of the Axiom, that two ſtraight lines cannot incloſe a ſpace. But, when the defects of this definition ſeem to be wholly corrected, we might certainly expect to be able to derive all the properties of ſtraight lines directly from that definition, and yet the fact is, that we are not at all aſſiſted by it, in the caſe before us. I confeſs this is not eaſily accounted for; but there are two conſiderations, which, though they may not contain the ſolution of the paradox, will perhaps ſerve to render it leſs wonderful.

The firſt is, that the definition given of an angle is certainly imperfect. An angle is the *inclination* of two lines; now, the word inclination is not much better underſtood than the word angle, ſo that we have here the very ſame defect that there is in Euclid's definition of a ſtraight line. It is, at the ſame time, difficult to conceive any way in which this definition can be amended; and it is, perhaps, on account of its imperfection that we are obliged to aſſume ſome property of the lines ſubtending an angle, or of two lines making angles with a third line, as an Axiom; juſt as the imperfect definition of a ſtraight line muſt be aſſiſted by the aſſumption, that two ſtraight lines cannot incloſe a ſpace.

It muſt farther be remarked, that whatever be the ſource of this difficulty, it is not the only one of the kind that we meet with in the Elements of Geometry. A ſecond inſtance occurs, where a certain relation between the lengths of ſtraight, and curve lines is aſſumed as an Axiom, without being logically deduced from our ideas, either of ſtraightneſs, or of curvature. This is the Axiom on which Archimedes, and all the geometers after him have founded the compariſon of the lengths of curves with the lengths of ſtraight lines, and is the ſame which is placed here at the beginning of the 8th Book. It would be in vain, I believe, that one would ſeek to give a rigorous demonſtration of that propoſition, yet it is of a nature purely elementary, though more complex, without doubt, than that which we have been conſidering. There is a third propoſition of this ſort, relative to ſurfaces, which is alſo laid down by Archimedes, for the foundation of the compariſon of curve ſuperficies with plane figures, and of which no demonſtration is given. Theſe are the only three roperties of geometrical magnitudes in the whole ſcience, hat are taken for granted without being deduced from the efinitions; that it is impoſſible to demonſtrate any of them, s what no one will take upon him to affirm; but the many nd powerful efforts made for that purpoſe, which they have lready withſtood, ought to deter any one from throwing away much of his time in ſearching after ſuch demonſtraions.

BOOK II. PROP. VII.

THE demonſtration of this propoſition uſually occaſions ſome difficulty to beginners, and on that account another is added, which is ſomewhat ſhorter.

PROP. A and B.

Theſe Theorems are added on account of their great uſe in geometry, and their cloſe connection with the other propoſitions which are the ſubject of this Book. Prop. A is an extenſion of the 9th and 10th.

BOOK III.

BOOK III.

DEFINITIONS.

THE definition which Euclid makes the firſt of this Book is that of equal circles, which he defines to be "thoſe "of which the diameters are equal." This is rejected from among the definitions, as being a Theorem, the truth of which is proved by ſuppoſing the circles applied to one another, ſo that their centres may coincide, for the whole of the one muſt then coincide with the whole of the other. The converſe, viz. That circles which are equal have equal diameters, is proved in the ſame way.

The definition of the angle of a ſegment is alſo omitted, becauſe it is not a rectilineal angle, but one underſtood to be contained between a ſtraight line and a portion of the circumference of a circle. For the ſame reaſon, no notice is taken in the 16th propoſition of the angle comprehended between the ſemicircle and diameter, which is ſaid by Euclid to be greater than any acute rectilineal angle. The reaſon for theſe omiſſions has already been aſſigned in the notes on the fifth definition of the firſt Book.

BOOK V.

Book V. THE ſubject of proportion has been treated ſo differently by thoſe who have written on elementary geometry, and the method which Euclid has followed has been ſo often, and ſo inconſiderately cenſured, that in theſe notes it will not perhaps be more neceſſary to account for the changes that I have made, than for thoſe that I have not made. The changes are but few, and relate to the language, not to the eſſence of the demonſtrations;

lemonſtrations; they will be explained after ſome of the definitions have been particularly conſidered.

DEF. III.

The definition of ratio given here has been greatly extolled by ſome authors; but whatever value it may have in the eyes of a metaphyſician, it has but little in thoſe of a geometer, becauſe nothing concerning the properties of ratios can be deduced from it. Dr Barrow has very judiciouſly remarked concerning it, "That Euclid had probably no other deſign "in making this definition, than to give a general ſummary "idea of ratio to beginners by premiſing this metaphyſical "definition, to the more accurate definitions of ratios that are "equal to one another, or one of which is greater or leſs than "the other: I call it a metaphyſical, for it is not properly a "mathematical definition, ſince nothing in mathematics depends on it, or is deduced, nor, as I judge, can be deduced "from it." (Barrow's Lectures, lect. 3.) Dr Simſon thinks the definition has been added by ſome unſkilful editor, but there is no ground for that ſuppoſition, other than what ariſes from the definition being of no uſe; we may, however, well enough imagine, that a certain idea of order and method induced Euclid to give ſome general definition of ratio, before he uſed the term in the definition of equal ratios.

DEF. IV.

This definition is a little altered in the expreſſion; Euclid has it, that "magnitudes are ſaid to have a ratio to one another, when the leſs can be multiplied ſo as to exceed the "greater."

DEF. V.

One of the chief obſtacles to the ready underſtanding of the 5th Book of Euclid is the difficulty that moſt people find of reconciling the idea of proportion which they have already acquired, with the account of it that is given in this definition. Our firſt ideas of proportion, or of proportionality, are got by trying to compare together the magnitudes of external bodies; and, though they be at firſt abundantly vague and incorrect, they are uſually rendered tolerably preciſe by the ſtudy of arithmetic; and we then learn to diſtinguiſh a particular relation, in point of magnitude, which being in any caſe the ſame between two things that it is between other two, we ſay of all the four, that they are proportionals.

The ſimpleſt form in which this relation can appear is, when the firſt contains the ſecond the ſame number of times that the third contains the fourth; or, as we may expreſs it, when the firſt is the ſame multiple of the ſecond that the third is of the fourth. From this ſimilitude of relations between the firſt two and the ſecond two, we call them proportionals, or ſay, that the firſt is to the ſecond as the third to the fourth.

Proceeding to generaliſe this idea, and taking number in the firſt place for the ſubject of compariſon, we ſoon come to fix on this as the general notion of proportional numbers, that when there are four numbers, ſuch that the quotient ariſing from dividing, according to the common rules of arithmetic, the firſt by the ſecond, is the ſame with the quotient that ariſes from dividing, in like manner, the third by the fourth, theſe numbers are proportionals, or, we ſay that the firſt is to the ſecond as the third to the fourth.

Now, as the operation of arithmetical diviſion applies as readily to any two magnitudes, of the ſame kind, as it does to two numbers, the notion of proportion thus obtained may be conſidered as perfectly general. For, in arithmetic, after finding how often the diviſor is contained in the dividend, we multiply the remainder by 10, or 100, or 1000, or any power, as it is called, of 10, and proceed to enquire how oft the diviſor is contained in this new dividend; and, if there be any remainder, we go on to multiply it by 10, 100, &c. as before, and to divide the product by the original diviſor, and ſo on, the diviſion ſometimes terminating, by no remainder being left, and ſometimes going on *ad infinitum*, in conſequence of a remainder being left at each operation. Now, this proceſs may eaſily be imitated with any two magnitudes A and B, providing they be of the ſame kind, or ſuch that the one can be multiplied ſo as to exceed the other. For, ſuppoſe that B is the leaſt of the two; take B out of A as oft as it can be found, and let the quotient be noted, and alſo the remainder, if there be any; multiply this remainder by 10, or 100, &c. ſo as to exceed B, and let B be taken as oft as it can be found out of the quantity produced by this multiplication, and let the quotient be noted, and alſo the remainder, if there be any. Proceed with this remainder as before, and ſo on continually; and it is evident, that we have an operation that applies to all magnitudes whatſoever, and that may be performed with reſpect to any two lines, any two plane figures, or any two ſolids, &c.

Now, when we have two magnitudes and two others, and find that the firſt divided by the ſecond, according to this method, gives the very ſame ſeries of quotients that the third does when divided by the fourth, we ſay of theſe magnitudes, as we did of the numbers above deſcribed, that the firſt is to the ſecond as the third to the fourth. There are only two more circumſtances to be taken notice of as neceſſary to give to this idea of proportion, the utmoſt generality of which it is capable.

Firſt, it is known from arithmetic, that the multiplication of the ſucceſſive remainders each of them by 10, is equivalent to multiplying the quantity to be divided by the product of all thoſe tens; ſo that multiplying, for inſtance, the firſt remainder by 10, the ſecond by 10, and the third by 10, is the ſame thing, with reſpect to the quotient, as if the quantity to be divided had been at firſt multiplied by 1000; and therefore, our ſtandard of the proportionality of numbers may be expreſſed thus: If the firſt multiplied any number of times by 10, and then divided by the ſecond, gives the ſame quotient as when the third is multiplied as often by 10 and then divided by the fourth, the four magnitudes are proportionals.

Again, it is evident, that there is no neceſſity in theſe multiplications for confining ourſelves to 10, or the powers of 10, and that we do ſo, in arithmetic, only for the conveniency of the decimal notation; we may therefore uſe any numbers whatſoever, providing we uſe the ſame in both caſes. Hence, we have this definition of proportionals, when there are four magnitudes, and any multiple whatſoever of the firſt, when divided by the ſecond, gives the ſame quotient

with

Book V.

with the like multiple of the third when divided by the fourth, the four magnitudes are proportionals, or the firſt has the ſame ratio to the ſecond that the third has to the fourth.

We are now arrived very near to Euclid's definition; for, let A, B, C, D be four proportionals, according to the definition juſt given, and m any number; and let the multiple of A by m, that is mA, be divided by B; and, firſt, let the quotient be the number n exactly, then alſo, when mC is divided by D, the quotient will be n exactly. But, when mA divided by B gives n for the quotient, $mA = nB$ by the nature of diviſion, ſo that when $mA = nB$, $mC = nD$, which is one of the conditions of Euclid's definition.

Again, when mA is divided by B, let the diviſion not be exactly performed, but let n be the quotient as far as it can be expreſſed in whole numbers, then $mA > nB$; and, for the ſame reaſon, $mC > nD$, which is another of the conditions of Euclid's definition.

Laſtly, Let us ſuppoſe, that mA does not contain B ſo often as there are units in n, then mA will be leſs than nB; but becauſe, by hypotheſis, mC does not contain D oftener than mA contains B, mC will not contain D ſo often as there are units in n, and therefore $mC < nD$. Therefore, we call A, B, C, D proportionals, when they are ſuch, that if $mA > nB$, $mC > nD$; if $mA = nB$, $mC = nD$; and if $mA < nB$, $mC < nD$, m and n being any numbers whatſoever. Now, this is exactly the criterion of proportionality eſtabliſhed by Euclid in the 5th definition, and is derived here by generaliſing the common and moſt familiar idea of proportion.

It appears from this, that the condition of mA containing B, whether with or without a remainder, as often as mC contains D, with or without a remainder, and of this being the caſe whatever value be aſſigned to the number m, includes in it all the three conditions that are mentioned in Euclid's definition; and hence, that definition may be expreſſed a little more ſimply by ſaying, that four magnitudes are proportionals, when any multiple of the firſt contains the ſecond, (with or without remainder), as oft as the ſame multiple of the third contains the fourth. But, though this definition is certainly, in the expreſſion, more ſimple than Euclid's, it is not, as will be found on trial, ſo eaſily applied to the purpoſe of demonſtration. The three conditions which Euclid brings together in his definition, though they ſomewhat embarraſs the impreſſion of it, have the advantage of rendering the demonſtrations more ſimple than they would otherwiſe be, by avoiding all diſcuſſion about the magnitude of the remainder left, after B is taken out of *m*A as oft as it can be found. All the attempts, indeed, to demonſtrate the properties of proportionals rigorouſly, by means of other definitions than Euclid's, either that I have made myſelf, or that I have ſeen others make, have only ſerved to convince me of the excellence of the method followed by the Greek geometer, and of his ſingular ſucceſs in generaliſing the idea of porportion.

The great objection to the other methods is, that if they are meant to be rigorous, they require two demonſtrations to every propoſition, one when the diviſion of *m*A into parts equal to B can be exactly performed, the other when it cannot be exactly performed, whatever value be aſſigned to *m*, or when A and B are what is called incommenſurable; and this laſt caſe will in general be found to require an indirect demonſtration, or a *reductio ad abſurdum*.

M. D'Alembert, ſpeaking of the doctrine of proportion, in a diſcourſe that contains many excellent obſervations, but in which he has overlooked Euclid's manner of treating this ſubject entirely, has the following remark: “On ne peut demontrer que de cette maniere, (la réduction à l'abſurde) la plupart des propoſitions qui regardent les incommenſurables. L'idee de l'infini entre au moins implicitement dans la notion de ces ſortes des quantités; et comme nous n'avons qu'une idee negative de l'infini on ne peut demontrer directement, *et a priori*, tout ce qui concerne l'infini mathematique.” (*Encyclopedie. Mot, Geometrie.*)

This remark ſets in a ſtrong and juſt light the difficulty of demonſtrating the propoſitions that regard the proportion of incommenſurable magnitudes, without having recourſe to the *reductio ad abſurdum*, but it is ſurpriſing that M. D'Alembert, a geometer no leſs learned than profound, ſhould have neglected to make mention of Euclid's method, the only one in which the difficulty he ſtates ſo ſtrongly is completely overcome. It is overcome by the introduction of the idea of indefinitude, (if I may be permitted to uſe the word), inſtead of the idea of infinity; for *m* and *n*, the multipliers employed, are ſuppoſed to be indefinite, or to admit of all poſſible values, and it is by the ſkilful uſe of this condition that the neceſſity of indirect demonſtrations is avoided. In the whole of geometry, I know not that any happier invention is to be found; and it is worth remarking, that Euclid appears in another of his works to have availed himſelf of the idea of indefinitude with the ſame ſucceſs, viz. in his books of Poriſms, which, have been reſtored by Dr Simſon, and in which the whole analyſis turned on that idea, as I have ſhewn at length, in the third volume of the Tranſactions of the Royal Society of Edinburgh. The inveſtigations of thoſe propoſitions were founded entirely on the principle of certain magnitudes admitting of innumerable values; and the methods of reaſoning concerning them ſeem to have been extremely ſimilar to thoſe employed in the fifth of the Elements. It is curious to remark this analogy between the different works of the ſame author; and to conſider, that the ſkill, in the conduct of this very refined and ingenious method, which Euclid had acquired in treating the properties of proportionals, may have enabled him to ſucceed ſo well in treating the ſtill more difficult ſubject of Poriſms.

With ſuch an opinion of Euclid's manner of treating proportion, as I have now expreſſed, it was impoſſible that I ſhould attempt to change any thing in the principle of his demonſtrations. I have only ſought to improve the language of them, by introducing a concise mode of expreſſion, of the ſame nature with that which we uſe in arithmetic, and in algebra. Ordinary language conveys the ideas of the different operations ſuppoſed to be performed in theſe demonſtrations ſo ſlowly, and breaks them down into ſo many parts, that they make not a ſufficient impreſſion on the underſtanding. This, indeed, will generally happen when the things treated of are not repreſented to the ſenſes by Diagrams, as they cannot be when we reaſon concerning magnitude in general, as in this part of the Elements. Here we ought certainly to adopt the language of arithmetic or algebra, which, by its ſhortneſs, and the rapidity with which it places objects before us, makes up in the beſt manner poſſible for being merely a conventional language, and uſing ſymbols that have no reſemblance to the things expreſſed by them. Such a language, therefore, I have endeavoured to introduce here; and, I am convinced, that if it ſhall be found an improvement, it is the only one of which the fifth of Euclid

id will admit. In other reſpects I have followed Dr Sim-
m's edition, to the accuracy of which it would be difficult to
ake any addition.

In one thing I muſt obſerve, that the doctrine of proportion,
laid down here, is meant to be more general than in Euclid's
lements. It is intended to include the properties of propor-
onal numbers as well as of all magnitudes. Euclid has not this
eſign, for he has given a definition of proportional numbers in
e ſeventh Book, very different from that of proportional
agnitudes in the fifth; and it is not eaſy to juſtify the logic
f this manner of proceeding; for we can never ſpeak of two
umbers and two magnitudes both having the ſame ratios,
nleſs the word ratio have in both caſes the ſame ſignifi-
ation. All the propoſitions about proportionals here given
re therefore underſtood to be applicable to numbers; and
ccordingly, in the eighth Book, the propoſition that proves
quiangular parallelograms to be in a ratio compounded of
e ratios of the numbers proportional to their ſides, is de-
onſtrated by help of the propoſitions of the fifth Book.

On account of this, the word quantity, rather than magni-
ude, ought in ſtrictneſs to have been uſed in the enunciation
f theſe propoſitions, becauſe we employ the word quantity
o denote, not only things extended, to which alone we give
e name of magnitudes, but alſo numbers. It will be ſuffi-
ient, however, to remark, that all the propoſitions reſpecting
e ratios of magnitudes relate equally to all things of which
ultiples can be taken, that is, to all that is uſually expreſſed
y the word quantity in its moſt extended ſignification, ta-
ing care always to obſerve, that ratio takes place only among
ke quantities. (See Def. 4.)

DEF. X.

The definition of compound ratio was firſt given accurate-
by Dr Simſon; for, though Euclid uſed the term, he did ſo
thout defining it. I have placed this definition before thoſe
duplicate and triplicate ratio, as it is in fact more general,
d as the relation of all the three definitions is beſt ſeen

when

Book V.
when they are ranged in this order, and expreſſed in the
ner done here ; duplicate ratio being called a ratio com
ed of two equal ratios, and triplicate of three equal ratio

It was juſtly obſerved by Dr Simſon, that the exp
compound ratio is introduced merely to prevent circumlo
and for the ſake principally of enunciating theſe prop
with conciſeneſs that are demonſtrated by reaſoning *ex*
that is, by reaſoning from the 22d or 23d of this Book.
will be evident to any one who conſiders carefully the P
of this, or the 23d of the 6th Book.

BOOK VI.

DEFINITION II.

Book VI.
THIS definition is changed from that of *reciprocal fi*
which was of no uſe, to one that correſponds t
language uſed in the 14th and 15th propoſitions, and in
parts of geometry.

PROP. XXVII. XXVIII. XXIX.

As conſiderable liberty has been taken with theſe pr
tions, it is neceſſary that the reaſons for doing ſo ſhould b
plained. In the firſt place, when the enunciations are
lated literally from the Greek, they ſound very harſhly
are, in fact, extremely obſcure. The phraſe of applyi
a ſtraight line, a parallelogram deficient, or exceeding by
ther parallelogram, is ſo elliptical and ſo little analog
ordinary language, that there could be no doubt of the
priety of at leaſt changing the enunciations.

It next occurred, that the problems themſelves in the 28th and 29th propoſitions are propoſed in a more general form than is neceſſary in an elementary work, and that, therefore, to take thoſe caſes of them that are the moſt uſeful, as they happen to be the moſt ſimple, muſt be the beſt way of accommodating them to the capacity of a learner. The problem which Euclid propoſes in the 28th is, "To a given ſtraight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram ſimilar to a given parallelogram;" which alſo might be more intelligibly enunciated thus: "To cut a given line, ſo that the parallelogram that has in it a given angle, and that is contained under one of the ſegments of the given line, and a ſtraight line which has a given ratio to the other ſegment, may be equal to a given ſpace;" inſtead of which problem I have ſubſtituted this other; "to divide a given ſtraight line ſo that the rectangle under its ſegments may be equal to a given ſpace." In the actual ſolution of problems, the greater generality of the former propoſition is an advantage more apparent than real, and is fully compenſated by the ſimplicity of the latter, to which, alſo, it is always eaſily reducible.

The ſame may be ſaid of the 29th, which Euclid enunciates thus: "To a given ſtraight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram ſimilar to a given parallelogram." This might be propoſed otherwiſe; "to produce a given line, ſo that the parallelogram having in it a given angle, and contained by the whole line produced, and a ſtraight line that has a given ratio to the part produced, may be equal to a given rectilineal figure." Inſtead of this, is given the following problem, more ſimple, and, as was obſerved in the former inſtance, very little leſs general: "To produce a given ſtraight line, ſo that the rectangle contained by the ſegments, between the extremities of the given line, and the point to which it is produced, may be equal to a given ſpace."

PROP. A, B, C, &c.

There are eight propoſitions added to this Book, on account of their utility and their connection with this part of the elements. The firſt four of them are in Dr Simson's edition, and among theſe Prop. A is given immediately after the third, being, in fact, a ſecond caſe of that propoſition, and capable of being included with it, in one enunciation. Prop. D is remarkable for being a theorem of Ptolemy the aſtronomer, in his Μεγαλη Συνταξις, and the foundation of the conſtruction of his trigonometrical tables. Prop. E is the ſimpleſt caſe of the former; it is alſo uſeful in trigonometry, and, under another form, was the 97th, or, in ſome editions, the 94th of Euclid's Data. The propoſitions F and G are very uſeful properties of the circle, and are taken from the *Loci Plani* of Apollonius. Prop. H is a very remarkable property of the triangle.

BOOK VII.

Book VII.

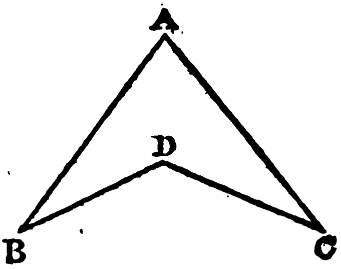
The reaſon for departing from Euclid in the geometry of ſolids has been already explained in the Preface, ſo that it only remains to make a few remarks on ſome particular definitions and theorems.

DEF. VIII. and PROP. XX.

Solid angles, which are defined here in the ſame manner as in Euclid, are magnitudes of a very peculiar kind, and may be remarked for not admitting of that accurate compariſon, one with another, which is common in the other ſubjects of geometry. It cannot, for example, be ſaid of one ſolid angle, that it is the half, or the double of another ſolid angle, nor did any geometer ever think of propoſing the problem of biſecting a given ſolid angle. In a word, no multiple or ſubmultiple of ſuch an angle can be taken, and we have no way of expounding, even in the ſimpleſt caſes, the ratio which one of them bears to another.

In this reſpect, therefore, a ſolid angle differs from every other magnitude that is the ſubject of mathematical reaſoning, all of which have this common property, that multiples and ſubmultiples of them may be found. It is not our buſineſs here to enquire into the reaſon of this anomaly, but it is plain, that on account of it, our knowledge of the nature and the properties of ſuch angles can never be very far extended, and that our reaſonings concerning them, muſt be chiefly confined to the relations of the plane angles, by which they are contained. One of the moſt remarkable of thoſe relations is that which is demonſtrated in the 20th of this Book, and which is, that all the plane angles which contain any ſolid angle muſt together be leſs than four right angles. This propoſition is the 21ſt of the 11th of Euclid.

This propoſition, however, is ſubject to a reſtriction in certain caſes, which, I believe, was firſt obſerved by M. le Sage of Geneva, in a communication to the Academy of Sciences of Paris in 1756. When the ſection of the pyramid formed by the planes that contain the ſolid angle, is a figure that has none of its angles exterior, ſuch as a triangle, a parallelogram, &c. the truth of the propoſition juſt enunciated cannot be queſtioned. But, when the aforeſaid ſection is a figure like that which is annexed, viz. ABDC, having ſome of its angles, ſuch as BDC, exterior, or, as they are ſometimes called, re-entering angles, the propoſition is not neceſſarily true; and it is plain, that in ſuch caſes the demonſtration which we have given, and which is the ſame with Euclid's, will no longer apply. Indeed, it were eaſy to ſhew, that on baſes of this kind, by multiplying the number of ſides, ſolid angles



Book VII.

angles may be formed, ſuch that the plane angles which contain them ſhall exceed four right angles by any quantity aſſigned. An illuſtration of this from the properties of the ſphere is perhaps the ſimpleſt of all others. Suppoſe that, on the ſurface of a hemiſphere there is deſcribed a figure, bounded by any number of arches of great circles making angles with one another, on oppoſite ſides alternately, the plane angles at the centre of the ſphere that ſtand on theſe arches may evidently exceed four right angles, and that, too, by multiplying and extending the arches, in any aſſigned ratio. Now, theſe plane angles contain a ſolid angle at the centre of the ſphere, according to the definition of a ſolid angle.

We are to underſtand the propoſition in the text, therefore, to be true only of thoſe ſolid angles in which the inclination of the plane angles are all the ſame way, or all directed toward the interior of the figure. To diſtinguiſh this claſs of ſolid angles from that to which the propoſition does not apply, it is perhaps beſt to make uſe of this criterion, that they are ſuch, that when any two points whatſoever are taken in the planes that contain the ſolid angle, the ſtraight line joining thoſe points falls wholly within the ſolid angle; or thus, they are ſuch, that a ſtraight line cannot meet the planes which contain them in more than two points. It is thus, too, that I would diſtinguiſh a plane figure that has none of its angles exterior, by ſaying, that it is a rectilineal figure, ſuch that a ſtraight line cannot meet the boundary of it in more than two points.

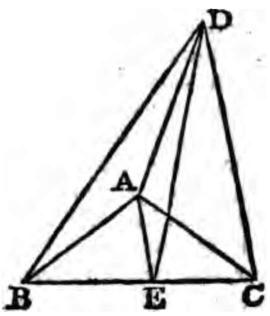
We diſtinguiſh, therefore, ſolid angles into two ſpecies, one in which the bounding planes can be interſected by a ſtraight line only in two points; and another where the bounding planes may be interſected by a ſtraight line in more than two points; to the firſt of theſe the propoſition in the text applies, to the ſecond it does not.

Whether Euclid meant entirely to exclude the conſideration of figures of the latter kind, in all that he has ſaid of ſolids, and of ſolid angles, it is not now eaſy to determine: It is certain, that his definitions involve no ſuch excluſion; and as the introduction of any limitation would conſiderably embarraſs theſe definitions, and render them difficult to be underſtood by a beginner, I have left it out, referring to this place the full explanation of the difficulty. I cannot conclude this note without remarking, with the hiſtorian of the

Academy, that it is extremely ſingular, that not one of all thoſe who had read or explained Euclid before M. le Sage, appears to have been ſenſible of this miſtake. (*Memoires de l'Acad. des Sciences* 1756, *Hiſt. p.* 77.) A circumſtance that renders this ſtill more ſingular is, that another miſtake of Euclid on the ſame ſubject, and perhaps of all other geometers, eſcaped M. le Sage alſo, and was firſt diſcovered by Dr Simſon, as will appear in the following note.

DEF. IX. and PROP. XXI.

These relate to ſimilar and equal ſolids, a ſubject on which miſtakes have prevailed not unlike to that which has juſt been mentioned. The equality of ſolids, it is natural to expect, muſt be proved like the equality of plane figures, by ſhewing that they may be made to coincide, or to occupy the ſame ſpace. But, though it be true that all ſolids which can be ſhewn to coincide are equal and ſimilar, yet it does not hold converſely, that all ſolids which are equal and ſimilar can be made to coincide. Though this aſſertion will, perhaps, at firſt, appear a paradox, yet the proof of it is extremely ſimple.



Let ABC be an iſoſceles triangle, of which the equal ſides are AB and AC; from A draw AE perpendicular to the baſe BC, and BC will be biſected in E. From E draw ED perpendicular to the plane ABC, and from D, any point in it, draw DA, DB, DC to the three angles of the triangle ABC. The pyramid DABC is divided into two pyramids DABE, DACE, which, though no one will diſpute their equality, cannot be ſo applied to one another as to coincide. For, though the triangles ABE, ACE are equal, BE being equal to CE, EA common to both, and the angles AEB, AEC equal, becauſe they are right angles, yet if theſe two triangles be applied to one another, ſo as to coincide, the ſolid DACE will, nevertheleſs, as is evident, fall without the ſolid DABE, for the two ſolids will be on the oppoſite ſides of the plane ABE. In the ſame way, though all the planes of the pyramid DABE may eaſily be ſhewn to be equal to thoſe of the pyramid DACE, each to each; yet will the pyramids themſelves never coincide, though the equal planes be applied to one another, becauſe they are on the oppoſite ſides of thoſe planes.

It may be ſaid, then, on what ground do we conclude the pyramids to be equal? The anſwer is, becauſe their conſtruction is entirely the ſame, and the conditions that determine the magnitude of the one, identical with thoſe that determine the magnitude of the other. For the magnitude of the pyramid DABE is determined by the magnitude of the triangle ABE, the length of the line ED, and the poſition of ED, in reſpect of the plane ABE; three circumſtances that are preciſely the ſame in the two pyramids, ſo that there is nothing that can determine one of them to be greater than another.

This reaſoning appears perfectly concluſive and ſatisfactory, and it ſeems alſo very certain, that there is no other principle on which the relation of the ſolids DABE, DACE to one another, can be determined. Neither is this a caſe that occurs rarely; it is one, that in the compariſon of magnitudes having three dimenſions, preſents itſelf continually; for, though two plane figures that are equal and ſimilar can always be made to coincide, yet, with regard to ſolids that are equal and ſimilar, if they have not a certain regularity in their conſtruction, there will be found juſt as many caſes in which they cannot, as in which they can coincide. Even figures deſcribed on ſurfaces, if they are not plane ſurfaces, may be equal and ſimilar without the poſſibility of coinciding. Thus, in the figure deſcribed on the ſurface of a ſphere, called a ſpherical triangle, if we ſuppoſe it to be iſoſceles, and a perpendicular to be drawn from the vertex on the baſe, it will not be doubted, that it is thus divided into two right angled ſpherical triangles equal and ſimilar to one another, and which, nevertheleſs, cannot be ſo laid on one another as to agree. The ſame holds in innumerable other inſtances, and therefore it is evident, that the Axioms at preſent admitted into Geometry are not ſufficient for the compariſon of all geometrical magnitudes, and that there is a principle, more general, and fundamental than that of the equality of coinciding figures, which ought to be introduced. What this principle is has alſo appeared very clearly in the courſe of theſe remarks; and it is, indeed, no other than the principle ſo celebrated in the philoſophy of Leibnitz, under the name of THE SUFFICIENT REASON. For it was ſhewn, that the pyramids DABE and DACE are concluded to be equal, becauſe each of them is determined to be of a certain magnitude, rather than of any other, by conditions that are the ſame in both, ſo that there is no REASON for the one being greater than the other. This Axiom may be rendered general by ſaying, That things of which the magnitude is determined by conditions that are exactly the ſame, are equal to one another; or, it might be expreſſed thus: Two magnitudes A and B are equal, when there is no reaſon that A ſhould exceed B, rather than that B ſhould exceed A. Either of theſe will ſerve as the fundamental principle for comparing geometrical magnitudes of every kind; they will apply in thoſe caſes where the coincidence of magnitudes with one another has no place; and they will apply with great readineſs, to the caſes in which a coincidence may take place, ſuch as in the 4th, the 8th, or the 26th of the Firſt Book of the Elements.

The only objection to this Axiom is, that it is ſomewhat of a metaphyſical kind, and belongs to the doctrine of the *ſufficient reaſon*, which is looked on with a ſuſpicious eye by ſome philoſophers. But this is no ſolid objection; for ſuch reaſoning may be applied with the greateſt ſafety to thoſe objects with the nature of which we are perfectly acquainted, and of which we have complete definitions, as in pure mathematics. In phyſical queſtions, the ſame principle cannot be applied with equal ſafety, becauſe in ſuch caſes we have ſeldom a complete definition of the things we reaſon about, or one that includes in it all their properties. Thus, when Archimedes proved the ſpherical figure of the earth, by reaſoning on a principle of this ſort, he was led to a falſe concluſion, becauſe he knew nothing of the rotation of the earth on its axis, which places the particles of that body, though at equal diſtances from the centre, in circumſtances very different from one another. But, concerning thoſe things that are the creatures of the mind altogether, like the objects of mathematical inveſtigation, there can be no danger of being miſled by the principle of the ſufficient reaſon, which at the ſame time furniſhes us with the only ſingle Axiom, by help of which we can compare together geometrical quantities, whether they be of one, of two, or of three dimenſions.

From the remarks juſt made, it neceſſarily follows, that ſome demonſtrations relating to the compariſon of ſolids that have hitherto been reckoned unexceptionable, are not concluſive, at leaſt, not ſo generally ſo, as the enunciations import. Such is the Prop. C, which Dr Simſon has introduced into the 11th of Euclid, with a view of ſupplying the defects of Euclid's method. The propoſition is, that "ſolid figures contained by the ſame number of equal and ſimilar planes, alike ſituated, and having none of their ſolid angles contained by more than three plane angles, are equal and ſimilar to one another." Dr Simſon proves the truth of this propoſition by placing the ſolids on the ſame baſe, and ſo making them to coincide. This, however, cannot always be done with ſolids that are ſimilar and equal, as has been ſhewn, ſo that the propoſition is neceſſarily limited in its extent, and that limitation ought not to be underſtood as implied in the words, alike ſituated. For this phraſe is applied to plane figures, in which, when they are ſimilar and equal, a coincidence may always be ſuppoſed; and it certainly ſignifies in that caſe nothing but that the ſides are placed in the ſame order, and that the angles they make with one another, are directed the ſame way with reſpect to the figure in each; that is, if any angle is exterior, or *re-entering* in the one, it is the ſame in the other, and ſo on. It is in the ſame way that it ſhould be underſtood when applied to ſolids.

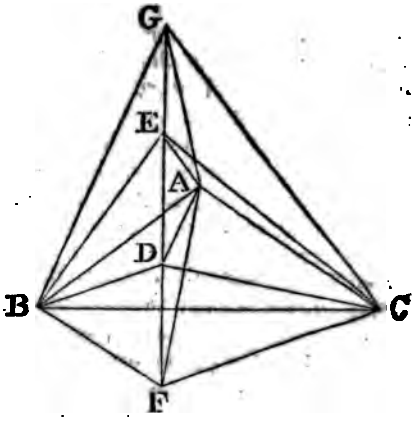
For the ſame reaſon, the 21ſt propoſition of the ſeventh Book of theſe Elements is not demonſtrated in its full extent. This, however, does not affect the ſubſequent reaſoning, becauſe the caſe which is demonſtrated is ſufficient for the proof of the propoſitions that follow.

Another remark concerning the geometry of ſolids is, that Euclid, and the mathematicians who followed him, were in the wrong, when they held, that thoſe ſolids are equal which are contained under the ſame number of equal and ſimilar planes, without adding, that the planes muſt be ſimilarly ſituated. For this remark we are indebted to Dr Simſon, who has ſhewn, that unleſs the condition, of the planes being ſimilarly ſituated, be added, the propoſition is falſe, in ſo much that innumerable ſolids may exiſt, contained by the ſame number of ſimilar and equal planes, that ſhall be all unequal to one another. He has proved the ſame with reſpect to ſolid angles, which were before held to be equal when they were contained by the ſame number of equal plane angles; for he ſhews that there, too, unleſs the plane angles be ſimilarly ſituated, the ſolid angles may be unequal.

This remark was publiſhed by Dr Simſon, in the firſt edition of his Euclid, in 1756, on the very ſame year when M. le Sage communicated to the Academy of Sciences the obſervation on the ſame ſubject mentioned in the laſt note; and it is ſingular, that theſe two geometers, without any communication with one another, ſhould, almoſt at the ſame time, have made two diſcoveries ſo nearly connected with one another, and yet, that neither of them ſhould have perceived the whole of the truth, ſo that the diſcovery of each is incomplete without that of the other.

When we ſpeak of the planes of two ſolids being ſimilarly ſituated, it is always meant, that the planes are in the ſame order, and their inclinations directed the ſame way in reſpect of the ſolid. Dr Simſon has proved the truth of his remark as follows:

“Let there be any plane rectilineal figure, as the triangle ABC, and from a point D within it, draw the ſtraight line DE at right angles to the plane ABC; in DE take DE, DF equal to one another, upon the oppoſite ſides of the plane, and let G be any point in EF; join DA, DB, DC; EA, EB, EC; FA, FB, FC; GA, GB, GC: Becauſe the ſtraight line EDF is at right angles to the plane ABC, it makes right angles with DA, DB, DC, which it meets in that plane; and in the triangles EDB, FDB, ED and DB are equal to FD and DB, each to each, and they contain right angles; therefore the baſe EB is equal to the baſe FB; in the ſame manner EA is equal to FA, and EC to FC: And in the triangles EBA, FBA, EB, BA are equal to FB, BA, and the baſe EA is equal to the baſe FA; wherefore the angle EBA is equal to the angle FBA, and the triangle EBA equal to the triangle FBA, and the other angles equal to the other angles; therefore theſe triangles are ſimilar: In the ſame manner the triangle EBC is ſimilar to the triangle FBC, and the triangle EAC to FAC; therefore there are two ſolid figures, each of which is contained by ſix triangles, one of them by three triangles, the common vertex of which is the point G, and their baſes the ſtraight lines AB, BC, CA, and by three other triangles the common vertex of which is the point E, and their baſes the ſame lines AB, BC, CA: The other ſolid is contained by the ſame three triangles, the common vertex of which is G, and their baſes AB, BC, CA; and by three other triangles, of which the common vertex is the point F, and their baſes the ſame ſtraight lines AB, BC, CA: Now the three triangles GAB, GBC, GCA are common to both ſolids, and the three others EAB, EBC, ECA of the firſt ſolid have been ſhown equal and ſimilar to the three others FAB, FBC, FCA of the other ſolid, each to each; therefore, theſe two ſolids are contained by the ſame number of equal and ſimilar planes: But that they are not equal is manifeſt, becauſe the firſt of them is contained in the other: Therefore it is not univerſally true, that ſolids are equal which are contained by the ſame number of equal and ſimilar planes."



"Cor. From this it appears, that two unequal ſolid angles may be contained by the ſame number of equal plane angles."

"For the ſolid angle at B, which is contained by the four plane angles EBA, EBC, GBA, GBC is not equal to the ſolid angle at the ſame point B, which is contained by the four plane angles FBA, FBC, GBA, GBC; for this laſt contains the other; And each of them is contained by four plane angles, which are equal to one another, each to each, or are the ſelf ſame, as has been proved: And indeed, there may be innumerable ſolid angles all unequal to one another, which are each of them contained by plane angles that are equal to one another, each to each: It is likewiſe manifeſt, that the before mentioned ſolids are not ſimilar, ſince their ſolid angles are not all equal."

PROP. XXXIII.

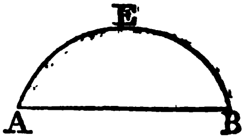
The principal departure from the method of Euclid, introduced into this Book, reſpects the pyramid; in treating of which, the method of exhauſtions is firſt employed, as in this propoſition. The demonſtration uſed here has the advantage of applying very generally to all the ſolids which require that method, and which can be treated of in the Elements; and it has alſo a very cloſe and evident affinity to the methods uſed in the higher geometry.

BOOK VIII.

Book VIII.

THE object of this Book being to compare with one another, and with the parallelepiped, the ſolids that depend on the circle, it is neceſſary to begin with the quadrature of that figure, or the compariſon of the ſpace contained within it with rectilineal figures. The firſt eight propoſitions relate entirely to this ſubject, and contain the rules both for finding the length of the circumference nearly, and alſo the ſpace contained within it, and in demonſtrating them, I have followed partly the method of Thomas Simpſon, in his Elements, and partly that of Archimedes, in his *Dimenſio circuli*. As the foundation of the propoſition for finding the

Book VIII. length of the circumference, an Axiom is prefixed to the beginning of this Book, the ſame in effect with that which Archimedes employs as the baſis of his inveſtigation, though ſomewhat more conciſely expreſſed. I have already taken notice of this Axiom, in the notes on parallel lines, (p. 372.) as a propoſition which it has been found impoſſible to demonſtrate, though it certainly might be expected to follow neceſſarily from the ideas of a ſtraight, and a curve line. Another thing remarkable concerning this Axiom is, that in the ſimpleſt caſe of it, when the figures are rectilineal, it admits of demonſtration; for it then coincides with the 20th and 21ſt of the firſt of Euclid, in the former of which it is ſhewn, that any two ſides of a triangle are greater than the third; and in the latter, that of two triangles which have the ſame baſe, that which is within the other, has the ſum of its ſides the leaſt. But when, inſtead of two ſides of a triangle, we have a curve AEB, and a ſtraight line AB terminated in the ſame points A and B, it then ceaſes to be poſſible to prove, that AEB is greater than AB; as alſo, that any curve line which includes AEB, and is terminated in A and B, is greater than AEB; and the truth of both theſe propoſitions muſt be taken for granted. At leaſt, after Archimedes has conſidered them as Axioms, we may reaſonably deſpair of ever ſeeing them demonſtrated. Here, therefore, we find ourſelves under the neceſſity of taking for granted a general propoſition, after having demonſtrated the ſimpleſt caſe of it. For all this, the trouble taken, to demonſtrate that caſe, ought not to be conſidered as ſuperfluous, for we are thereby prepared to admit the general Axiom, and have the ſatisfaction of obſerving its perfect agreement with truths, previouſly demonſtrated. It is probably from the circumſtance of a curve line having no definition, but one which is merely negative, viz. that it is a line of which no part is a ſtraight line, that the impoſſibility of demonſtrating this propoſition takes its riſe.



PROP.

PROP. IV. V. &c.

The demonſtrations of the 5th and 6th propoſitions require the method of exhauſtions, that is to ſay, they prove a certain property to belong to the circle, becauſe it belongs to the rectilineal figures inſcribed in it, or deſcribed about it according to a certain law, in the caſe when thoſe figures approach to the circle ſo nearly as not to fall ſhort of it, or to exceed it by any aſſignable difference. This principle is general, and is the only one by which we can poſſibly compare curvelineal, with rectilineal ſpaces, or the length of curve lines with the length of ſtraight lines, whether we follow the methods of the ancient or of the modern geometers. It is, therefore, a great injuſtice to the latter methods to repreſent them as ſtanding on a foundation leſs ſecure than the former; they ſtand in reality on the ſame, and the only difference is, that the application of the principle, common to them both, is more general and expeditious in the one caſe than in the other. This identity of principle, and affinity of the methods uſed in the elementary and the higher mathematics, it ſeems the more neceſſary to obſerve, that ſome learned mathematicians have appeared not to be ſufficiently aware of it, and have even endeavoured to demonſtrate the contrary. An inſtance of this is to be met with in the preface of the valuable edition of the works of Archimedes, lately printed at Oxford. In that preface, Torelli, the learned commentator, whoſe labours have done ſo much to elucidate the writings of the Greek geometer, but who is ſo unwilling to acknowledge the merit of the modern analyſis, undertakes to prove, that it is impoſſible from the relation which the rectilineal figures inſcribed in, and circumſcribed about, a given curve, have to one another, to conclude any thing concerning the properties of the curvelineal ſpace itſelf, except in certain circumſtances which he has not preciſely deſcribed. With this view he attempts to ſhew, that if we are to reaſon from the relation which certain rectilineal figures belonging to the circle have to one another, notwithſtanding that thoſe figures may approach ſo near to the circular ſpaces within which they are inſcribed, as not to differ from them by any aſſignable magnitude, we ſhall be led into error, and ſhall ſeem to prove, that the circle is to the ſquare of its diameter exactly as 3 to 4. Now, as this is a concluſion which the diſcoveries of Archimedes himſelf, prove ſo clearly to be falſe, Torelli argues, that the principle from which it is deduced muſt be falſe alſo; and in this he would no doubt be right, if his former concluſion had been fairly drawn. But the truth is, that a very groſs paralogiſm is to be found in that part of his reaſoning, where he makes a tranſition from the ratios of the ſmall rectangles inſcribed in the circular ſpaces, to the ratios of the ſums of thoſe rectangles, or of the whole rectilineal figures. In doing this, he takes for granted a propoſition which, it is wonderful, that one who had ſtudied geometry in the ſchool of Archimedes, ſhould for a moment have ſuppoſed to be true. The propoſition taken in the ſimpleſt view of it is this: If A, B, C, D, E, F, be any number of magnitudes, and a, b, c, d, e, f, as many others; and if $A : B :: a : b$,

$C : D :: c : d$,

$E : F :: e : f$, then the ſum of A, C and E will be to the ſum of B, D and F, as the ſum of a, c and e, to the ſum of b, d and f; or $A+C+E : B+D+F :: a+c+e : b+d+f$. Now, this propoſition, which Torelli ſuppoſes to be perfectly general, is not true, except in two caſes, viz. either firſt, when $A : C :: a : c$, and

$A : E :: a : e$; and conſequently,

$B : D :: b : d$, and

$B : F :: b : f$; or, ſecondly, when all the ratios of A to B, C to D, E to F, &c. are equal to one another. To demonſtrate this, let us ſuppoſe that there are four magnitudes, and four others,

thus, $A : B :: a : b$, and

$C : D :: c : d$, then we cannot have $A+C : B+D :: a+c : b+d$, unleſs either, $A : C :: a : c$, and $B : D :: b : d$; or $A : C :: b : d$, and conſequently $a : b :: c : d$.

Take a magnitude K, ſuch that a : c : : A : K, and another L, ſuch that b : d : : B : L; and ſuppoſe it true, that A+C : B+D : : a+c : b+d. Then, becauſe by inverſion, K : A : : c : a, and, by hypotheſis, A : B : : a : b, and alſo B : L : : b : d, *ex æquo*, K : L : : c : d; and conſequently, K : L : : C : D.

K,	A,	B,	L.
c,	a,	b,	d.

Again, becauſe A : K : : a : c, by addition,
A+K : K : : a+c : c; and, for the ſame reaſon,
B+L : L : : b+d : d, or, by inverſion,
L : B+L : : d : b+d. And, ſince it has been ſhewn, that K : L : : c : d; therefore, *ex æquo*,

A+K,	K,	L,	B+L.
a+c,	c,	d,	b+d.

A+K : B+L : : a+c : b+d; but by hypotheſis,
A+C : B+D : : a+c : b+d, therefore
A+K : A+C : : B+L : B+D.

Now, firſt, let K and C be ſuppoſed equal, then, it is evident, that L and D are alſo equal; and therefore, ſince by conſtruction a : c : : A : K, we have alſo a : c : : A : C; and, for the ſame reaſon, b : d : : B : D, and theſe analogies form the firſt of the two conditions, of which one is affirmed above to be always eſſential to the truth of Torelli's propoſition.

Next, if K be greater than C, then, ſince
A+K : A+C : : B+L : B+D, by diviſion,
A+K : K—C : : B+L : L—D. But, as was ſhewn,
K : L : : C : D, by converſion and alternation,
K—C : K : : L—D : L, therefore, *ex æquo*,
A+K : K : : B+L : L, and, laſtly, by diviſion,
A : K : : B : L, or A : B : : K : L, that is,
A : B : : C : D.

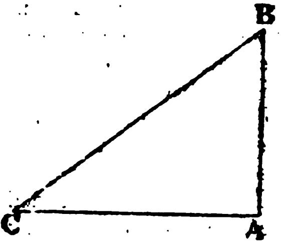
Wherefore, in this caſe the ratio of A to B is equal to that of C to D, and conſequently, the ratio of a to b equal to that of c to d. The ſame may be ſhewn, if K is leſs than G; therefore, in every caſe there are conditions neceſſary to the truth of Torelli's propoſition, which he does not take into account, and which, as is eaſily ſhewn, do not belong to the magnitudes to which he applies it.

In conſequence of this, the concluſion that is meant to be eſtabliſhed reſpecting the circle falls entirely to the ground, and with it the more general inference that was aimed againſt the modern analyſis.

It will not, I hope, be imagined, that I have taken notice of theſe circumſtances with any deſign to leſſen the reputation of the learned Italian, who has in ſo many reſpects deſerved well of the mathematical ſciences, or to detract from the value of a poſthumous work, which, by its elegance and correctneſs, does ſo much honour to the Engliſh editors. But I would warn the ſtudent againſt that ſpirit of party, which ſeeks to introduce itſelf even into the inveſtigations of geometry, and to perſuade us, that elegance, and even truth, are qualities poſſeſſed excluſively by the ancient methods of demonſtration. The high tone in which Torelli cenſures the modern mathematics, is the more calculated to produce an opinion of this kind, that it is aſſumed by one who had ſtudied the writings of Archimedes with ſo much diligence; and therefore, the obſervations, that have been made above, may be uſeful, by teaching us to liſten with caution to his deciſions.

PROP. VII.

This propoſition is the foundation of the application of arithmetic to geometry, and may be ſaid to be the truth that connects together theſe two branches of the mathematics, though, in elementary treatiſes, I think, it has for the moſt part been omitted. In no caſe do we compute, in numbers, the area of any figure, or even calculate the length of a ſide of a triangle, from the other parts which determine it being given, without having recourſe to this Theorem. If, for inſtance, from having two ſides of a right angled triangle as ABC expreſſed in numbers, we would compute the length of f the remaining ſide BC, we muſt make uſe of this propoſition, as well as of the 47th of the firſt Book. If, for inſtance, AB=6, and AC=8, and, if it be required to find BC; then, ſince AB is to AC as 6 to 8, the ſquare of AB will be to the ſquare of AC, by this propoſition, as 6×6 to 8×8, or as 36 to 64, and therefore, alſo the ſum of the ſquares on AB and AC will be to the ſquare of AB as 100 to 36. But the ſquares of AB and AC are equal to the ſquare of BC; therefore, the ſquare of BC is to the ſquare of AB as 100 to 36, and therefore, by the ſecond corollary, BC is to BA as 10 to 6. It is the ſame in other caſes; and, though the ſteps may not all be taken ſo regularly as is done here, they are nevertheleſs as certainly implied.



PROP. VIII.

This enunciation is the ſame with that of the third of the *Dimenſio Circuli* of Archimedes; but the demonſtration is different, though it proceeds, like that of the Greek geometer, by the continual biſection of the 6th part of the circumference. In this propoſition a particular notation is uſed, by putting the ſign + after a number, to denote that ſomething is to be added to it, and the ſign —, to denote the contrary. Thus, when it is ſaid, that AQ is to AD as 866.0254+ to 1000, this ſignifies, that AQ is to AD as a number greater than 866.0254 is to 1000, and it is the ſame thing with ſaying, that AQ has to AD a greater ratio than 866.0254 to 1000. In the ſame manner, when it is demonſtrated in the ſecond part of the propoſition, that LN is to CD as 32.71927— to 1000, it is meant, that LN is to CD as a number leſs than 32.71927 to 1000, or that LN has to CD a leſs ratio than that of 32.71927 to 1000.

The advantage that reſults from this mode of expreſſion is, that when we are to reaſon about the ſums or differences of two lines, of which the ratio is thus expreſſed, it can be done by

Book VIII.

by the ſimple addition or diviſion of proportionals, witho
any new propoſition. Thus, when it is ſhewn that AQ,
which is the ſame, DH is to DA as 866.0254+ to 1000, i
evident from the 18th of the 5th, that DH and DA to
ther, thas is, AH is to DA as 1866.0254+ to 1000; wher
if it had been ſaid that DH is to DA in a greater ratio th
866.0254 to 1000, a ſeparate propoſition, or *lemma*, muſt ha
been introduced, for the expreſs purpoſe of proving that H
has to DA a greater ratio than 1866.0254 to 1000.

It muſt alſo be attended to, in reading this propoſition, th
the ſecond figure is a part of the firſt, repreſented on a lar
ſcale, for the ſake of rendering the ſmall diviſions of the c
cumference, and the lines belonging to them, more diſtinct.

FINIS.